

From Euler numbers to the golden ratio: rigidity, ranges and the most ambiguous Parikh matrices for words with repeated letters

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Abstract

The degree of ambiguity of a Parikh matrix, in Salomaa’s sense, is the number of words that share it. For permutations the adjacent pair counts determine the matrix, and the most ambiguous matrices are the two alternating ones, each shared by an Euler number of words (classical facts on descent sets, where the author’s 2006 thesis stopped). With repeated letters a fibre of the pair counts can split into several Parikh classes. For every content we characterise the pair-count vectors that determine the whole matrix: the condition is local, one of four arithmetic rules on every three consecutive letters. For three letters we find the least and greatest triple count given both pair counts; the least remainder in the Cauchy inequality of Mateescu, Salomaa and Yu is $\min(rs, (b-r)(b-s))$, where r and s are the residues of the pair counts modulo the middle multiplicity b . When the triple count is not forced, its two greatest values differ by two exactly on an arithmetic family and two corner families, and by one elsewhere. For content (a, b, a) every largest pair-count fibre lies on the antidiagonal $x + y = ab$, and for $(2, b, 2)$ with $b \geq 16$ the largest Parikh class has $\lfloor (b^2 + 9b + 36)/14 \rfloor$ words. For $(a, 2, c)$ with $a \leq c$ the largest Parikh class has $\binom{a+c}{a}$ words if and only if $c(c-1) \leq a(a+c-1)$, a threshold asymptotic to the golden ratio, with equality exactly at $(F_n F_{n+1}, F_n F_{n+2})$, n odd, a classical family; the largest pair-count fibre is the centre, uniquely unless $a = c = 1$. For k letters each occurring n times, a total-positivity argument, with exact counts for $n \leq 3$, shows that the alternating classes are never the largest when $n \geq 4$ and $k \geq 3$, or $n = 3$ and $k \geq 4$, while for $n = 2$ every class differing from them at one end loses; we conjecture that for $n = 2$ they are the unique largest classes once $k \geq 6$, as verified for $k \leq 12$.

Keywords: Parikh matrix; M-ambiguity; scattered subword; multiset permutation; Euler numbers; total positivity.

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1 Introduction

Ambiguity of Parikh matrices. Let $1 < 2 < \dots < k$ be an ordered alphabet. A word w has *content* $\mathbf{n} = (n_1, \dots, n_k)$ if each letter i occurs $n_i \geq 1$ times in it; $|w|_u$ is the number of occurrences of u in w as a scattered subword, and $|w|_{j..l} = |w|_{j(j+1)\dots l}$. The Parikh matrix $\Psi(w)$ [21] is the upper unitriangular $(k+1) \times (k+1)$ matrix with entries $|w|_{r..s-1}$, $r < s$; its second diagonal is the content and its third the vector of pair counts $\psi_2(w) = (|w|_{12}, \dots, |w|_{(k-1)k})$. Words with the same Parikh matrix are matrix equivalent, or M-equivalent [19, Definition 2, p. 4] (the report version of [20]; see also [31, Definition 3, p. 225]) and form a *Parikh class*. Salomaa calls the number $d(\Psi)$ of words with Parikh matrix Ψ its degree of ambiguity (and the degree of M-ambiguity of each of these words), “a very natural notion not at all studied by now”, and shows that it has no polynomial bound in the length [34, Definition 6, p. 109, and Theorem 8, p. 111]. We study the largest degree of ambiguity $d_{\max}(\mathbf{n})$ at content $\mathbf{n}$, and the most ambiguous matrices, which attain it.

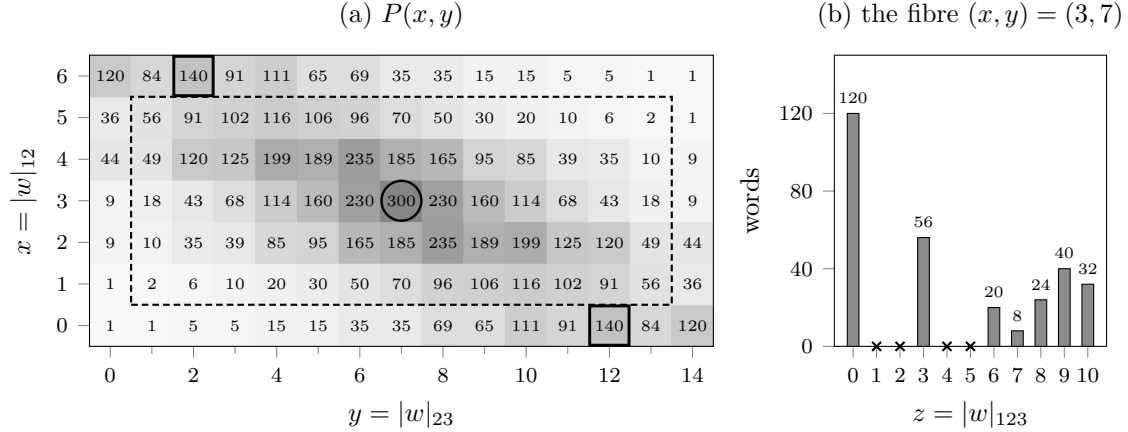


Figure 1: Content $(3, 2, 7)$. (a) The ψ_2 -fibre sizes $P(x, y)$ for $0 \leq x \leq 6$ and $0 \leq y \leq 14$ (darker is larger). Every cell outside the dashed frame, on the edges of the box, is a single Parikh class. The circled centre $(3, 7)$ is the unique largest ψ_2 -fibre, with 300 words (Theorem G). The two boxed cells carry the largest Parikh classes, $(x, y, z) = (0, 12, 0)$ and $(6, 2, 6)$, with 140 words each, more than $\tau = \binom{10}{3} = 120$ (Theorem F). (b) The central fibre splits by $z = |w|_{123}$ into seven classes, of sizes 120, 56, 20, 8, 24, 40, 32 at $z = 0, 3, 6, 7, 8, 9, 10$; the crosses mark the missing values $z = 1, 2, 4, 5$.

Permutations. For the permutations of $[k]$, of content (1^k) , ψ_2 determines the Parikh matrix [48, Theorem 3.5, printed pp. 19–20, PDF pp. 23–24], and $\psi_2(w)$ is the descent vector of w^{-1} , with 0 for a descent and 1 for an ascent, so the class sizes are the numbers of permutations with a given descent set [48, Theorems 6.12 and 6.15, printed pp. 31–34, PDF pp. 35–38]. By a theorem of Niven and de Bruijn (see [42, Corollary 1.6.5]), $d_{\max}(1^k)$ is the Euler number E_k , attained for $k \geq 2$ exactly by the two alternating classes, $\psi_2(w) = (1, 0, 1, \dots)$ and $(0, 1, 0, \dots)$. These are translations of classical facts on descent sets, and the thesis does not name the Euler numbers; its Computation 5.2 [48, printed p. 27, PDF p. 31] lists the class sizes for $k = 5$, with the maximum $16 = E_5$ at $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$, and the research project of Yamamura [47] announces the same maximum for permutations. The thesis ends by asking for “a generalized formula for the Parikh vector statistic”, the number of words with a given ψ_2 , when letters repeat [48, §8, printed p. 37, PDF p. 41]; for three letters, Theorems D and G locate the largest of these numbers.

Size against splitting. With repeated letters ψ_2 no longer determines the matrix. At fixed content the words with $\psi_2(w) = \mathbf{x}$ form the ψ_2 -fibre $\psi_2^{-1}(\mathbf{x})$, a union of Parikh classes, so $d(\Psi) \leq |\psi_2^{-1}(\mathbf{x})|$ when $\mathbf{x}$ is the third diagonal of Ψ . A class is large only if its fibre is large and splits little, and the two pull against each other: for three letters the fibres on the boundary of the box of pair counts are single classes, while the large fibres, near its centre, split into many. In Figure 1, for $(3, 2, 7)$, the centre $(x, y) = (3, 7)$ is the largest fibre, with 300 words, but it splits into seven classes, the largest of 120 words, and the largest classes, with 140 words each, are whole fibres on the boundary. We ask when a fibre does not split (*rigidity*, Section 3), how it splits for three letters (*range*, Section 4), and which fibres and classes are largest (*size*, Sections 5 to 7); Table 1 summarises the answers.

Rigidity. Let $\gamma(m, n) = \{0, 1, mn - 1, mn\}$ if $m, n \geq 2$, and $\{0, 1, \dots, mn\}$ otherwise.

Theorem A (Rigidity). Let $k \geq 3$ and $0 \leq x_i \leq n_i n_{i+1}$ for every i . The fibre $\psi_2^{-1}(\mathbf{x})$ is a single Parikh class if and only if, for every $i \leq k - 2$, with $(a, b, c) = (n_i, n_{i+1}, n_{i+2})$, one of

the following rules holds:

$$\begin{aligned} \text{(E)} \quad & x_i \in \{0, ab\} \text{ or } x_{i+1} \in \{0, bc\}; \quad \text{(N)} \quad x_i + x_{i+1} \leq b; \\ \text{(Ch)} \quad & (ab - x_i) + (bc - x_{i+1}) \leq b; \quad \text{(\Gamma)} \quad x_i \in \gamma(a, b) \text{ and } x_{i+1} \in \gamma(b, c). \end{aligned}$$

So ψ_2 determines the Parikh matrix on content $\mathbf{n}$ if and only if $\mathbf{n}$ is isolated: $\min(n_i, n_{i+1}) = 1$ for every i . The number Rig_k of single-class ψ_2 -fibres of (2^k) satisfies $\sum_{k \geq 1} \text{Rig}_k t^k = t(1-t)(1+2t)/(1-4t-2t^2+4t^3)$, with $\text{Rig}_1 = 1$ and $\text{Rig}_2 = 5$ counting all vectors; thus $\text{Rig}_3, \dots, \text{Rig}_6 = 20, 86, 364, 1548$.

Sufficiency is known or immediate: (Γ) is Salomaa's γ -property (Lemma 2.6), (Ch) for $k = 3$ the corner chamber of [49, Theorem 2], (N) its image under reversal, and (E) trivial. We claim the converse, the locality and the count, which were not located; [49, Theorem 6] and [19, §7, Theorem 8] contain special cases of the converse.

Range. For three letters write (a, b, c) for the content and $x = |w|_{12}$, $y = |w|_{23}$, $z = |w|_{123}$. The fibre of (x, y) splits into one class for each z in the set $\mathcal{Z}(x, y)$ of its values, the corner entries compatible with the rest of the matrix, for which Mateescu, Salomaa and Yu “can offer no systematic method” [22, §8, p. 19]. They proved $\lambda \leq z$ and the Cauchy inequality $bz \leq xy$, where $\lambda = cx + ay - abc$; the remainders count subwords: $\Delta = xy - bz = |w|_{2132} + |w|_{2312}$ and $z - \lambda = |w|_{321}$ (Lemma 2.5). Write $x = bq_x + r$, $y = bq_y + s$ with $0 \leq r, s < b$, and $x = aQ + R$, $y = cT + S$ with $0 \leq R < a$, $0 \leq S < c$; $[\cdot]$ is the Iverson bracket.

Theorem B (Exact range). *For $0 \leq x \leq ab$ and $0 \leq y \leq bc$, the least values of the two remainders on the fibre of (x, y) are*

$$\begin{aligned} \min \Delta &= \min(rs, (b-r)(b-s)), \\ \min |w|_{321} &= \max(0, -\lambda) + [Q + T = b - 1] \cdot \min(RS, (a-R)(c-S)). \end{aligned}$$

Hence xy/b is attained if and only if b divides x or y , a direct consequence of the equality case of [22, Lemma 2], and $\lfloor xy/b \rfloor - z_{\max} \leq \lfloor b/4 \rfloor$. The branch $Q + T \geq b$, where $\min z = \lambda$, is [49, Lemma 4, eq. (4.1)]; the other extrema were not located.

Theorem C (The top of a fibre). *Let $b \geq 2$, and suppose that z takes at least two values on the fibre of (x, y) . Then $z_{\max} - 1 \notin \mathcal{Z}(x, y)$ if and only if (T2) $r = s = 0$, $1 \leq q_x \leq a - 1$ and $1 \leq q_y \leq c - 1$; or (T5) $2 \leq x \leq b - 1$ and $y = bc - x$; or (T5^R) $2 \leq y \leq b - 1$ and $x = ab - y$. In these cases $z_{\max} - 2 \in \mathcal{Z}(x, y)$.*

So the two largest values of z differ by two exactly on an arithmetic family and two corner families, and by one elsewhere. Two instances of (T2) are printed in [19, §7, Theorem 8], and next to the chamber of [49] the conclusion follows from [49, Theorem 6]; the general statement was not located.

Size: the contents (a, b, a) and $(2, b, 2)$. Let $P(x, y) = |\psi_2^{-1}(x, y)|$.

Theorem D (Largest pair-count fibres). *If $a = c$, then $P(x, y)^2 \leq P(x, ab - x)P(ab - y, y)$, with equality if and only if $x + y = ab$; hence every largest ψ_2 -fibre of (a, b, a) lies on the antidiagonal. For $(2, b, 2)$ the largest ψ_2 -fibres have $(h + 2)(h + 3)$ words, where $h = \lfloor b/2 \rfloor$: they are (b, b) alone if b is even, and exactly $(b - 1, b + 1)$ and $(b + 1, b - 1)$ if b is odd.*

Through the standard joint law of two Mann–Whitney statistics, the non-strict inequality, which already puts some largest fibre on the antidiagonal, is the positive semidefiniteness of i.i.d. mixtures; the strict form is a direct consequence, recorded but not claimed (Remark 5.3), and the $(2, b, 2)$ fibre sizes off the pairs covered by [49, Theorems 2 and 6] were not located.

Theorem E (Largest Parikh classes of $(2, b, 2)$). *For every $b \geq 16$, $d_{\max}(2, b, 2) = \lfloor (b^2 + 9b + 36)/14 \rfloor$. For $b \geq 20$, every largest Parikh class either has $z = 0$ and even pair counts with $|x - y| \leq 2$ and $|x + y - 8(b + 1)/7| < 3/2$, or is the image under reversal of such a class; there are two, four or six largest classes.*

Theorem E was not located; its proof is a lattice-point computation in the gap encoding of [49, Lemma 1] and [22, §7], with an exhaustive computation for $16 \leq b \leq 19$. With the values for $b \leq 15$, the formula holds exactly for $b = 10$, $b = 13$ and $b \geq 16$.

Size: two middle letters. For $(a, 2, c)$ the classes $(0, 2c, 0)$, $(2a, 0, 0)$ and $(a, c, 0)$ consist of the words $22u$, $u22$ and $2u2$, where u has a letters 1 and c letters 3; so $d_{\max}(a, 2, c) \geq \tau = \binom{a+c}{a}$ (cf. [34, pp. 110–111], with no letter 2 and $a = c$). No *trace class*, a set of words with prescribed subwords on all pairs of adjacent letters, is larger, as its words are determined by their subwords on $\{1, 3\}$.

Theorem F (The golden-ratio threshold). *Let $1 \leq a \leq c$. The largest Parikh class of $(a, 2, c)$ has exactly τ words if $c(c-1) \leq a(a+c-1)$, that is, if $c \leq c^*(a) = (a+1+\sqrt{5a^2-2a+1})/2$, and more otherwise; in every case it has $\max_{0 \leq j \leq c} \sum_{i=j}^{\min(2j,c)} \binom{a+c-i}{a}$ words, the size of the largest ψ_2 -fibre on the boundary of the box. Moreover $c^*(a) - \varphi a \in ((1-1/\sqrt{5})/2, (3-\sqrt{5})/2]$, where $\varphi = (1+\sqrt{5})/2$, and $c(c-1) = a(a+c-1)$ holds exactly for $(a, c) = (F_n F_{n+1}, F_n F_{n+2})$ with n odd; for these contents exactly five classes have τ words.*

So $c^*(a)/a \rightarrow \varphi$; the ties begin with $(1, 2)$, $(6, 10)$, $(40, 65)$, $(273, 442)$ and $(1870, 3026)$ (Figure 7), and $(3, 2, 7)$ has classes of $140 > \tau = 120$ words. The tie set is classical: by Corollary 6.5 the tie equation says $\binom{a+c-2}{a} = \binom{a+c-1}{a-1}$, which for $a \geq 2$ Singmaster solved through a Pell equation [39, §2] (see also [27, 26]). The rest of Theorem F was not located; its proof uses an exact finite certificate (Appendix A).

Theorem G (The unique largest ψ_2 -fibre). *If $(a, c) \neq (1, 1)$, the centre (a, c) is the unique largest ψ_2 -fibre of $(a, 2, c)$, and*

$$P(a, c) = \left(\left\lfloor \frac{a+c}{2} \right\rfloor + 1 \right) \binom{\lfloor (a+c)/2 \rfloor}{\lfloor a/2 \rfloor} \binom{\lceil (a+c)/2 \rceil}{\lceil a/2 \rceil}.$$

For $(a, c) = (1, 1)$ the three antidiagonal points tie. The closed form is an identity of Andrews and Paule [1, eq. (2.2)]; we re-prove it and claim only that the centre is a largest fibre, uniquely for $(a, c) \neq (1, 1)$, which was not located. Theorem G does not extend to every b : uniqueness fails for $b = 1$ (Remark 5.4), for odd b at $(2, b, 2)$ (Theorem D), and for $b = 6$ and $b = 10$; the largest fibres of $(7, 6, 7)$ are $(20, 22)$ and $(22, 20)$, with 484 422 words against 484 020 at the centre $(21, 21)$ (Proposition 5.5). Uniqueness holds for b even, $(a, c) \neq (1, 1)$ and $a + b + c \leq 19$ (verified by computer); whether it holds for every $b \equiv 0 \pmod{4}$ is part of Question 5.6.

Size: uniform contents. Let Z_k be the alternating class of (n^k) , each letter occurring n times, with $\psi_2 = (n^2, 0, n^2, 0, \dots)$. For a nonempty set $\mathcal{S}$ of words of content (n, n) over $\{1, 2\}$, without $1^n 2^n$, let $Z_k(\mathcal{S})$ be the set of words with subword on $\{1, 2\}$ in $\mathcal{S}$ and with the other adjacent-pair subwords of Z_k , and $r_k = |Z_k(\mathcal{S})|/|Z_k|$. A *one-end relaxation* of an alternating class, the ψ_2 -fibre obtained by changing its pair count at one end edge, has the size of some $Z_k(\mathcal{S})$.

Theorem H (Uniform contents). (i) *Let $n = 2$ and $k \geq 2$. In the central fibre $\psi_2 = (2, \dots, 2)$ of (2^k) , every entry $|w|_{j..l}$ with $l - j \geq 2$ is 0 or 2, and the fibre consists of exactly F_k Parikh classes.*

Table 1: Largest Parikh classes, and largest ψ_2 -fibres where marked ψ_2 ; “central” means a class of the central fibre, and τ and h are as above.

Content	Largest classes or fibres	Status
(1^k) , $k \geq 2$	alternating, E_k words each	classical [42, Corollary 1.6.5]
isolated, $k \geq 4$	alternating, and no others (Remark 7.1)	direct consequence of [14]
$(a, 2, c)$, $a \leq c$	τ words if $c(c-1) \leq a(a+c-1)$, more otherwise	Theorem F
$(a, 2, c)$	ψ_2 : the centre (a, c) , unique if $(a, c) \neq (1, 1)$	Theorem G
(a, b, a)	ψ_2 : all on the antidiagonal $x + y = ab$	direct consequence (Remark 5.3)
$(2, b, 2)$	ψ_2 : $(h+2)(h+3)$ words	Theorem D
$(2, b, 2)$, $b \geq 16$	$\lfloor (b^2 + 9b + 36)/14 \rfloor$ words	Theorem E
(n^k) , $n \geq 4$, $k \geq 3$	never alternating	Theorem H
(3^k) , $k \geq 4$	never alternating	Theorem H
(2^k)	central for $k = 4, 5$ ($56 > 53$, $844 > 840$); alternating, and no others, for $6 \leq k \leq 12$	verified (Proposition 7.28); Conjecture 7.29 for $k \geq 6$
$n_i \in \{1, 2\}$, $k \geq 4$	alternating, and no others, unless $\mathbf{n}$ begins with $(2, 2, 1)$, ends with $(1, 2, 2)$, or is (2^4) or (2^5)	Conjecture 7.30; one direction proved, verified for $4 \leq k \leq 11$

(ii) Let $n \geq 1$. For every such $\mathcal{S}$, $r_3 < r_5 < r_7 < \dots < r_6 < r_4 < r_2$.

(iii) Let $k \geq 3$. If $n \geq 4$, or $n = 3$ and $k \geq 4$, no alternating class of (n^k) is a largest class. If $n = 2$, every one-end relaxation of an alternating class has at most $\frac{50}{53}|Z_k|$ words.

Part (i) is a direct consequence of Simpson’s results on palindromes [38, Theorem 3(c),(d) and Corollary 4, p. 331], as each trace class of the central fibre contains one; only the explicit description of the classes and their count were not located. For $n = 1$, (ii) is a direct consequence of [5] (Remark 7.16); our proof of (ii) is a transfer-operator argument of a known kind, and (ii) for $n \geq 2$ and (iii) were not located. For $n = 2$, a $(2, 2, 1)$ end beats alternation for every tail (Proposition 7.3), and for (2^k) alternation loses for $k = 4, 5$ but wins uniquely for $6 \leq k \leq 12$ (Proposition 7.28, verified by computer); we conjecture that it wins for every $k \geq 6$ (Conjecture 7.29).

Credits and limits. Each result is credited in its section, with locators; “not located” means not found in the inspected sources cited there, and is our strongest priority claim. The author’s preprint [49] studies ternary words near the maximum abc of the triple count, jointly with inversions, with exact class sizes on the chamber $(ab - x) + (bc - y) \leq b$ and on the first exterior layer, where this sum is $b + 1$, and the attainable triple counts near abc ; we treat the extremes of z on every fibre, the largest fibres and classes, and every alphabet length. Table 2 lists the results of [49] used here and our few contacts with its further directions [49, §7], which we do not take up. Salomaa’s papers on the injectivity of Parikh matrix mappings [30], known to us only through [31, 4, 37], and on a Cauchy-type inequality [33], whose abstract announces a study of its equality cases, were not available to us, and [36, 44, 45, 13] were seen only as abstracts or metadata; these limit the search for Theorems A, B, C and F and for Lemma 3.2.

Verification and organisation. Every computational statement is recomputed by a standard-library Python check, distributed with the article as supplementary files (Section 8); proofs that use an exact finite computation (Theorems E and F and Theorem H(iii)) name it, and statements resting on computation alone are marked as verified, with their range. Section 2 fixes notation, Sections 3 to 7 prove Theorems A, B–C, D–E, F–G and H, Section 9 collects open problems, and Appendix A proves the main lemma of Section 6; the acknowledgements disclose the use of AI tools.

2 Preliminaries

This section fixes the language for classes and fibres and records the symmetries and, for three letters, the profile model; all of it is known or elementary. Table 2 lists the points of contact with the author's article [49], none of which is claimed. Letters $a < b < c$ of the sources are written $1 < 2 < 3$, and [4, 37, 15] are cited in the versions named in the bibliography.

2.1 Words, Parikh matrices and fibres

Definition 2.1. Let $k \geq 2$ and let $\mathbf{n} = (n_1, \dots, n_k)$ be a *content*, a vector of positive integers, with $|\mathbf{n}| = n_1 + \dots + n_k$. $\mathcal{W}(\mathbf{n})$ is the set of the $|\mathbf{n}|!/(n_1! \dots n_k!)$ words over $1 < 2 < \dots < k$ with n_i letters i . Subwords are scattered: $|w|_u$ counts the position sequences $p_1 < \dots < p_\ell$ with $w_{p_1} \dots w_{p_\ell} = u$; we write $|w|_{j..l} = |w|_{j(j+1)\dots l}$, so $|w|_{m..m} = n_m$, and put $|w|_{m..m-1} = 1$.

- (i) The *Parikh matrix* $\Psi(w)$ is the upper unitriangular $(k+1) \times (k+1)$ matrix with $\Psi(w)_{r,s} = |w|_{r..s-1}$ for $r \leq s$. Its second diagonal is $\mathbf{n}$; its third is the vector of *pair counts* $\psi_2(w) = \mathbf{x} = (x_1, \dots, x_{k-1})$, $x_i = |w|_{i(i+1)}$. *Edge* i joins the letters i and $i+1$; the *box* is $\prod_i \{0, \dots, n_i n_{i+1}\}$.
- (ii) The ψ_2 -*fibre* (pair-count fibre) of $\mathbf{x}$ is $\psi_2^{-1}(\mathbf{x}) = \{w \in \mathcal{W}(\mathbf{n}) : \psi_2(w) = \mathbf{x}\}$. A *Parikh class* is the set of words of $\mathcal{W}(\mathbf{n})$ with a given Parikh matrix Ψ ; its size $d(\Psi)$ is the degree of ambiguity of Ψ , or of M-ambiguity of its words [34, Definition 6, p. 109], and $d_{\max}(\mathbf{n})$ is the largest $d(\Psi)$ at content $\mathbf{n}$.
- (iii) The *projection* of w on edge i , its subword on the letters $i, i+1$, is encoded by $A^{(i)}(w) = (A_1^{(i)}, \dots, A_{n_{i+1}}^{(i)})$, where $A_t^{(i)}$ is the number of letters i before the t -th letter $i+1$. It lies in $\mathcal{A}_i(x_i)$, where $\mathcal{A}_i(u)$ is the set of nondecreasing sequences in $\{0, \dots, n_i\}^{n_{i+1}}$ with sum u , the *area*. A *trace class* is the set of words of $\mathcal{W}(\mathbf{n})$ with given projections.

Equivalently, $\Psi(w)$ is the product, from left to right, of the matrices $I + E_{m,m+1}$ over the letters m of w , where the $E_{r,s}$ are the matrix units [21, Theorem 2.1]; in the thesis it is $\psi_{M_k}(w)$ [48, Definition 2.8, Theorem 2.10, printed pp. 12–13, PDF pp. 16–17]. Let $\mathbf{B}_i(w)$ be the $n_i \times n_{i+1}$ matrix with (r, s) entry 1 if the r -th letter i of w precedes its s -th letter $i+1$ (that is, if $r \leq A_s^{(i)}$) and 0 otherwise, and let $\mathbf{1}$ be the all-ones column vector. In the expanded product below, the term indexed by $(r_j, \dots, r_l)$ is 1 exactly when the r_j -th letter j , $\dots$, the r_l -th letter l form an occurrence of $j(j+1)\dots l$. Hence

$$|w|_{j..l} = \mathbf{1}^\top \mathbf{B}_j(w) \mathbf{B}_{j+1}(w) \dots \mathbf{B}_{l-1}(w) \mathbf{1} \quad (1 \leq j < l \leq k). \quad (2.1)$$

So $\Psi(w)$ is a function of the projections, and classes and fibres are unions of trace classes. By [4, p. 7], the words of a trace class are those obtained from each other by commuting adjacent letters i, j with $|i - j| \geq 2$, called trivially M-equivalent there and stated to be M-equivalent; so trace classes are the traces of the partially commutative monoid with these commutations (see also the listing method of [19, §4, pp. 6–7]).

Lemma 2.2 (Realisation). *If $A^{(i)} \in \mathcal{A}_i(u_i)$ for $1 \leq i \leq k-1$, some word of $\mathcal{W}(\mathbf{n})$ has projection $A^{(i)}$ on every edge i . Hence, for $\mathbf{x}$ in the box, the projections map $\psi_2^{-1}(\mathbf{x})$ onto $\prod_i \mathcal{A}_i(x_i)$; in particular $\psi_2^{-1}(\mathbf{x}) \neq \emptyset$.*

Proof. Start from 1^{n_1} . When the letters $1, \dots, i$ are placed with the prescribed projections on the edges $1, \dots, i-1$, insert the t -th letter $i+1$ just before the $(A_t^{(i)} + 1)$ -th letter i , or at the end if $A_t^{(i)} = n_i$, in the order of t at each place. As $A^{(i)}$ is nondecreasing, the t -th inserted letter is the t -th letter $i+1$ and has $A_t^{(i)}$ letters i before it; earlier projections do not change. The rest follows, as $\mathcal{A}_i(u) \neq \emptyset$ for $0 \leq u \leq n_i n_{i+1}$. (Nonemptiness is also [19, Lemma 1, p. 3], the report version of [20], and for one edge [22, Lemma 8, p. 19].) $\square$

Edge i is *extreme* in $\mathbf{x}$ if $x_i \in \{0, n_i n_{i+1}\}$; then $\mathcal{A}_i(x_i)$ is a single sequence and $\mathbf{B}_i \in \{0, \mathbf{11}^\top\}$. A letter m with $n_m = 1$ is a *cut letter*. The *alternating classes* $Z(\mathbf{n})$ and $Z'(\mathbf{n})$ are the fibres of the two vectors with all edges extreme, alternately full ($x_i = n_i n_{i+1}$) and empty ($x_i = 0$), edge 1 being full in $Z(\mathbf{n})$; each is one trace class and, by (2.1), one Parikh class. We write (n^k) for the content $n_1 = \dots = n_k = n$, whose alternating classes Z_k, Z'_k have a common size $|Z_k|$ (Lemma 2.3). Further, $[n] = \{1, \dots, n\}$; $[S]$ is 1 if the statement S holds and 0 otherwise; $(u)_+ = \max(u, 0)$; v^m in a vector means m consecutive entries v ; $\binom{n}{m} = 0$ unless $0 \leq m \leq n$; E_n counts the alternating permutations of $[n]$; F_n and L_n are the Fibonacci and Lucas numbers ($F_1 = F_2 = 1, L_1 = 1, L_2 = 3$); $\varphi = (1 + \sqrt{5})/2$; and $\mathbb{E}$ is expectation.

2.2 Symmetries

Lemma 2.3 (Symmetries; [22, Theorem 3, p. 7], [37, Theorem 2.2, ms. p. 4]). *Let $w = w_1 \dots w_N \in \mathcal{W}(\mathbf{n})$ and $\mathbf{n}^{\text{rev}} = (n_k, \dots, n_1)$. The reversal $w^R = w_N \dots w_1$ lies in $\mathcal{W}(\mathbf{n})$; the complement w^C (each letter i replaced by $k + 1 - i$) and the reverse-complement w^{RC} (the complement of w^R) lie in $\mathcal{W}(\mathbf{n}^{\text{rev}})$. For $1 \leq j \leq l \leq k$,*

$$|w^R|_{j..l} = |w|_{l(l-1)\dots j}, \quad |w^C|_{j..l} = |w^R|_{(k+1-l)..(k+1-j)}, \quad |w^{\text{RC}}|_{j..l} = |w|_{(k+1-l)..(k+1-j)};$$

so $|w^R|_{i(i+1)} = n_i n_{i+1} - x_i$ and $\psi_2(w^{\text{RC}}) = (x_{k-1}, \dots, x_1)$. The matrices $\Psi(w^R), \Psi(w^C), \Psi(w^{\text{RC}})$ are functions of $\Psi(w)$; each map sends Parikh classes, fibres, trace classes and alternating classes to sets of the same kind and size, and reversal swaps $Z(\mathbf{n})$ and $Z'(\mathbf{n})$.

Proof. Reversing positions, renaming letters, or both, gives the formulas, and each pair of a letter i and a letter $i+1$ is in increasing order in exactly one of w and w^R . Expanding $\Psi(w)^{-1} = (I - E_{w_N, w_N+1}) \dots (I - E_{w_1, w_1+1})$, a product over the letters of w^R , gives $(\Psi(w)^{-1})_{j,l+1} = (-1)^{l-j+1} |w^R|_{j..l}$; this is [22, Definition 3, Theorem 3, p. 7], from [21]. So $\Psi(w^R)$, and by the formulas $\Psi(w^C)$ and $\Psi(w^{\text{RC}})$, are functions of $\Psi(w)$. The maps are bijections, with inverses of the same kind; they send projections to reversed or renamed ones and extreme edges to extreme edges, keeping adjacency, and reversal swaps full and empty edges. $\square$

We call $w \mapsto w^{\text{RC}}$ the *mirror map*; statements about $\mathbf{n}$ have mirror statements about $\mathbf{n}^{\text{rev}}$.

2.3 Three letters: profiles and gaps

For a ternary content (a, b, c) and $w \in \mathcal{W}(a, b, c)$ put $x = |w|_{12}, y = |w|_{23}, z = |w|_{123}$, the entries of $\Psi(w)$ above its second diagonal, and $x' = ab - x, y' = bc - y$ (the j and i of [49]); the pairs with $x' + y' = b + d$ form the *layer of index d , exterior* if $d \geq 1$; those with $x' + y' \leq b$ form the *chamber* of [49, Theorem 2]. A Parikh class is a triple (x, y, z) , of size $\rho(x, y, z)$, and a fibre a pair (x, y) , of size $P(x, y)$; the framed table collects the notation. Of the abc triples of a 1, a 2 and a 3 of w , cx have the 1 before the 2, ay the 2 before the 3, z both, and $|w|_{321}$ neither; so $|w|_{321} = abc - cx - ay + z$, and by Lemma 2.3, with $\lambda = cx + ay - abc$,

$$|w|_{321} = z - \lambda, \quad (|w^R|_{12}, |w^R|_{23}, |w^R|_{123}) = (x', y', z - \lambda). \quad (2.2)$$

So reversal maps the fibre of (x, y) onto that of (x', y') , with $\rho(x', y', z - \lambda) = \rho(x, y, z)$ and $\mathcal{Z}(x', y') = \mathcal{Z}(x, y) - \lambda$, where $\lambda = \lambda(x, y) = -\lambda(x', y')$; the complement maps (x, y, z) to $(y', x', z - \lambda)$ and the mirror map to (y, x, z) , for the content (c, b, a) . The identity $z - |w|_{321} = \lambda$ is [49, eq. (2.8), p. 3] and entry (1, 4) of [22, Theorem 3, p. 7], and follows from [22, Theorem 1, p. 4].

Lemma 2.4 (Profiles and gaps; [22, §7, pp. 15–16], [49, Lemma 1, eq. (4.3)]). *Let $w \in \mathcal{W}(a, b, c)$. For $1 \leq t \leq b$ let A_t and C_t be the numbers of letters 1 before and of letters 3 after*

Notation for a ternary content (a, b, c) , with letters $1 < 2 < 3$.			
x, y, z	$ w _{12}, w _{23}, w _{123}$	x', y'	$ab - x, bc - y$ (j, i in [49])
λ, Δ	$cx + ay - abc$ (2.2), $xy - bz$ (2.3)	ρ, P	the sizes of a class and of a fibre
$\mathcal{Z}(x, y)$	the values of z on the fibre; $z_{\max}, z_{\min}$	$q_x, r; q_y, s$	$x = bq_x + r, y = bq_y + s, 0 \leq r, s < b$
A_t, C_t	numbers of 1s before, 3s after the t -th 2	$A_{\leq \ell}, C_{> \ell}$	$A_1 + \dots + A_\ell; C_{\ell+1} + \dots + C_b$
$\mathcal{A}(x), \mathcal{C}(y)$	profiles with sums x, y (Lemma 2.4)	$W(A, C)$	$W(\alpha, \gamma) = \prod_g \binom{\alpha_g + \gamma_g}{\alpha_g}$
α_g, γ_g	numbers of 1s and 3s in gap g , at level $b - g$ (the number of 2s to their right)	$Q, R; T, S$	$x = aQ + R, y = cT + S; 0 \leq R < a, 0 \leq S < c$
p_i, q_j	2s before the i -th 1, after the j -th 3	$\gamma(m, n)$	see Lemma 2.6

the t -th letter 2 (the profiles A, C of w), and for $0 \leq g \leq b$ let α_g and γ_g be the numbers of letters 1 and 3 in gap g , the factor between the g -th and $(g+1)$ -th letter 2 (gap 0 precedes the first 2, gap b follows the last). Then A is nondecreasing in $\{0, \dots, a\}^b$, C is nonincreasing in $\{0, \dots, c\}^b$, $A_t = \alpha_0 + \dots + \alpha_{t-1}$, $C_t = \gamma_t + \dots + \gamma_b$, and

$$x = \sum_t A_t = ab - \sum_g g \alpha_g, \quad y = \sum_t C_t = \sum_g g \gamma_g, \quad z = A \cdot C = \sum_{g < h} (h - g) \alpha_g \gamma_h,$$

$$|w|_{321} = \sum_t (a - A_t)(c - C_t) = \sum_{g > h} (g - h) \alpha_g \gamma_h.$$

Conversely, every such pair (A, C) , equivalently every pair $\alpha, \gamma \in \mathbb{Z}_{\geq 0}^{b+1}$ with sums a and c , is the profile pair of exactly $W(A, C) = W(\alpha, \gamma) \geq 1$ words, which form one trace class. Hence, with $\mathcal{A}(x)$ and $\mathcal{C}(y)$ the sets of such A with sum x and such C with sum y , $\mathcal{Z}(x, y) = \{A \cdot C : A \in \mathcal{A}(x), C \in \mathcal{C}(y)\}$, $\rho(x, y, z) = \sum_{A \cdot C = z} W(A, C)$ and $P(x, y) = \sum_{A \in \mathcal{A}(x), C \in \mathcal{C}(y)} W(A, C)$.

Proof. A later 2 has at least as many 1s before it and at most as many 3s after it. An occurrence of 12, 23, 123 or 321 is a 2, say the t -th, with a 1 before it, a 3 after it, both, or a 3 before and a 1 after it: $A_t, C_t, A_t C_t$ or $(c - C_t)(a - A_t)$ choices. The gaps $0, \dots, t-1$ precede the t -th 2; a 1 or 3 in gap g has g letters 2 before it; and a 1 in gap g and a 3 in gap h have $|h - g|$ letters 2 between them, standing as $1 \dots 3$ if $g < h$ and as $3 \dots 1$ if $g > h$. Conversely, the differences $\alpha_0 = A_1$, $\alpha_g = A_{g+1} - A_g$, $\alpha_b = a - A_b$ and $\gamma_0 = c - C_1$, $\gamma_g = C_g - C_{g+1}$, $\gamma_b = C_b$ ($0 < g < b$) invert these relations, and a word with gap occupancies (α, γ) is fixed by the arrangement of its α_g letters 1 and γ_g letters 3 in each gap, chosen independently in $\binom{\alpha_g + \gamma_g}{\alpha_g}$ ways. Finally, A is the projection on edge 1, and C determines that on edge 2. $\square$

The profile identities are in the proof of the Cauchy inequality [22, §7, p. 15], with monotonicity for a one-letter middle word on p. 16 (also [19, §5, p. 8]); the profiles and their realisability are [49, eq. (4.3), p. 8]; and the gap form with W is [49, Lemma 1, eqs. (2.5)–(2.7), p. 3]: with p_i and q_j (its p_r, q_s) the numbers of 2s before the i -th 1 and after the j -th 3, its occupancies are our $\alpha_g = \#\{i : p_i = g\}$ and $\gamma_g = \#\{j : q_j = b - g\}$, and $z = \sum_{i,j} (b - p_i - q_j)_+$ (see also [32, §5, p. 1824]). Reversal maps $A_t, C_t, (\alpha_g, \gamma_g)$ to $a - A_{b+1-t}, c - C_{b+1-t}, (\alpha_{b-g}, \gamma_{b-g})$; the complement exchanges α and γ .

Lemma 2.5 (The two bounds; [22, Lemma 2, p. 12; Theorem 4, p. 9; §7, p. 15]). *For every $w \in \mathcal{W}(a, b, c)$, $\lambda \leq z \leq xy/b$, the second inequality being the Cauchy inequality of [22]. Moreover $z - \lambda = |w|_{321}$, and Δ satisfies the covariance identity*

$$\Delta = xy - bz = |w|_{2132} + |w|_{2312} = \sum_{1 \leq t < t' \leq b} (A_{t'} - A_t)(C_t - C_{t'}). \quad (2.3)$$

So $z = \lambda$ if and only if w avoids 321, and $z = xy/b$ if and only if w avoids 2132 and 2312, if and only if A or C is constant.

Table 2: Points of contact with [49] (version 1; PDF pages). Each item is cited where it is used.

Item of [49]	Use in this article
Lemma 1, eqs. (2.5)–(2.7), p. 3; eq. (4.3), p. 8	The gap encoding with the count W , and the profiles (Lemma 2.4); the lists p_i, q_j in Section 5
Eq. (2.8), p. 3	$z - w _{321} = \lambda$: (2.2), Lemma 2.5, and Lemma 3.2(v)
Theorem 2, p. 5, and the remark after it, p. 6	$z = \lambda$ on the chamber $x' + y' \leq b$: the sufficiency of (Ch) for $k = 3$ and, by reversal, of (N); Theorem A(i) gives the largest region where z is forced
Lemma 4, eq. (4.1), p. 8	λ is attained if and only if $Q + T \geq b$: this branch of Theorem B(ii) (Corollary 4.4); only the other branches and the gap are claimed
Theorem 6, eq. (5.1), p. 11	Exactly the values $\lambda, \lambda + 1$ on the layer of index 1 if $x', y', a, c \geq 2$: a case of the converse in Theorem A(i) and of Theorem C (Remark 4.12)
Theorems 2 and 6 at $a = c = 2$	Totals of Theorem D(iii) on $x + y \geq 3b$, on $x + y = 3b - 1$ with $x, y \leq 2b - 2$ and their reversals; classes $z = 0$ of Lemma 5.11 for $x + y \leq b$ and for $x + y = b + 1, x, y \geq 2$
Proposition 2, eq. (4.7), p. 11; Fig. 3, Table 1, p. 12	The values of z on $\mathcal{W}(2, b, 2)$, and three complete $(2, 3, 2)$ fibres; set beside Proposition 4.13 and Theorem D(iii), which fix (x, y)
Theorem 2 at (n, n, n)	The sizes $ Z_3(\mathcal{S}_u) $, $u \leq n$, in Corollary 7.7, by reversal: a trivial overlap
§7, item 2, p. 14	Fibre polynomials on the layers of index ≥ 2 , that article's further direction, are not given in general; we give some top coefficients there (Proposition 4.10; at $x = 2$ or $y = 2$ its multiplicity of $z_{\max} - 2$ is also the bottom coefficient) and, for $(2, b, 2)$, the values at (b, b) (Proposition 4.13) and the totals (Theorem D(iii)); the bound for thin classes, with $\rho(b, b, b)$, and the other bottom coefficients ($ w _{321} = 0$) are used only to prove Theorem E (Lemmas 5.10 and 5.11 and Remark 5.12)

Proof. The first identity is (2.2). By Lemma 2.4, $xy - bz = \sum_{t,t'} A_t(C_{t'} - C_t)$; symmetrising in t, t' gives the last form in (2.3), whose term (t, t') counts the pairs of a 1 and a 3 between the t -th and the t' -th 2, that is, the occurrences of 2132 and 2312 through these two 2s. The terms are nonnegative, and all vanish if and only if $A_1 = A_b$ or $C_1 = C_b$. $\square$

In [22, Lemma 2] the equality case reads “neither $bacb$ nor $bcab$ ”, and the first identity in (2.3) is in the displays on p. 11 there; [4, Lemma 3, p. 8] restates both bounds.

Lemma 2.6 (Unique projections). *Let $\gamma(m, n) = \{0, \dots, mn\}$ if $\min(m, n) \leq 1$, and $\gamma(m, n) = \{0, 1, mn - 1, mn\}$ otherwise. For every edge i and $0 \leq u \leq n_i n_{i+1}$, $|\mathcal{A}_i(u)| = 1$ if and only if $u \in \gamma(n_i, n_{i+1})$. In particular $|\mathcal{A}(x)| = 1$ if and only if $x \in \gamma(a, b)$, and $|\mathcal{C}(y)| = 1$ if and only if $y \in \gamma(b, c)$.*

Proof. We treat $\mathcal{A}(x) = \mathcal{A}_1(x)$; the general case is a renaming, and reversing C turns $\mathcal{C}(y)$ into the set $\mathcal{A}(y)$ of the pair (c, b) , with $\gamma(c, b) = \gamma(b, c)$. If $b = 1$ or $a = 1$, the profile (x) or $(0^{b-x}, 1^x)$ is unique. Let $a, b \geq 2$. For $x \in \{0, 1\}$ only 0^b and $(0^{b-1}, 1)$ occur, and $A \mapsto (a - A_b, \dots, a - A_1)$ maps $\mathcal{A}(x)$ onto $\mathcal{A}(ab - x)$. For $2 \leq x \leq ab - 2$, both $(q_x^{b-r}, (q_x + 1)^r)$ and $(0^{b-Q-1}, R, a^Q)$ lie in $\mathcal{A}(x)$, as $q_x < a$ and $Q < b$. The second has at most one entry strictly between 0 and a ; the first has at least two: all b entries if $1 \leq q_x \leq a - 2$, the $b - r \geq 2$ entries $a - 1$ if $q_x = a - 1$, and the $r = x \geq 2$ entries 1 if $q_x = 0$. So the two differ. $\square$

This is classical ($|\mathcal{A}(x)|$ is the coefficient of q^{ab-x} in $\begin{bmatrix} a+b \\ a \end{bmatrix}_q$ [42, Proposition 1.7.1]); for binary Parikh matrices it is [37, Theorem 2.4, ms. p. 5], attributed there to [30]. With (2.1) it gives Salomaa's γ -property [30], as quoted in [31, Theorem 3, p. 226] and [4, Theorem 3, p. 7]: if $x_i \in \gamma(n_i, n_{i+1})$ for all i , then $\psi_2^{-1}(\mathbf{x})$ is one trace class, so one Parikh class (rule (Γ) of Definition 3.5).

Two results of [49] are only set beside ours: Corollary 2, p. 5 (for $a \neq c$, z and the number of inversions, if at most b , determine $\Psi(w)$), next to Theorem A, and Theorem 7, p. 12 (the least $abc - z$ over words containing 321), next to the fibrewise least $|w|_{321}$ of Theorem B(ii).

3 Rigidity: when the pair counts determine the matrix

A Parikh class with matrix Ψ and pair counts $\mathbf{x}$ lies inside $\psi_2^{-1}(\mathbf{x})$, so $d(\Psi) \leq |\psi_2^{-1}(\mathbf{x})|$. When is this an equality, that is, when do the second and third diagonals determine the Parikh matrix? This is Salomaa's number inference problem [31, p. 225] for the higher diagonals, asked for each value of the pair counts. The known sufficient conditions are the γ -property (Lemma 2.6) and, for three letters, the chamber of [49, Theorem 2]; Theorem A gives the exact condition, and it is local. We write $j \cdots l$ for the word $j(j+1) \cdots l$.

Lemma 3.1 (Cut lemma). *Let w have content $\mathbf{n}$ and $\psi_2(w) = \mathbf{x}$, and let $1 \leq j < l \leq k$.*

- (i) *If $x_i = 0$ for some i with $j \leq i < l$, then $|w|_{j..l} = 0$.*
- (ii) *If $x_i = n_i n_{i+1}$ for some i with $j \leq i < l$, then $|w|_{j..l} = |w|_{j..i} \cdot |w|_{i+1..l}$.*
- (iii) *If $n_m = 1$ for some m with $j < m < l$, then $|w|_{j..l} = |w|_{j..m} \cdot |w|_{m..l}$.*

Proof. By (2.1), $x_i = \mathbf{1}^\top \mathbf{B}_i \mathbf{1}$ counts the ones of $\mathbf{B}_i$, so $\mathbf{B}_i$ is 0 if $x_i = 0$ and $\mathbf{1}\mathbf{1}^\top$ if $x_i = n_i n_{i+1}$; then $\mathbf{1}^\top \mathbf{B}_j \cdots \mathbf{B}_{i-1} (\mathbf{1}\mathbf{1}^\top) \mathbf{B}_{i+1} \cdots \mathbf{B}_{l-1} \mathbf{1}$ is the product of $\mathbf{1}^\top \mathbf{B}_j \cdots \mathbf{B}_{i-1} \mathbf{1} = |w|_{j..i}$ and $\mathbf{1}^\top \mathbf{B}_{i+1} \cdots \mathbf{B}_{l-1} \mathbf{1} = |w|_{i+1..l}$ (empty products being identities). If $n_m = 1$, the row $\mathbf{1}^\top \mathbf{B}_j \cdots \mathbf{B}_{m-1}$ and the column $\mathbf{B}_m \cdots \mathbf{B}_{l-1} \mathbf{1}$ are the numbers $|w|_{j..m}$ and $|w|_{m..l}$. $\square$

Parts (i) and (ii) are immediate from (2.1); part (iii) with $l - j = 2$ is, after renaming the letters, the instance $b = 1$ of the equality case of the Cauchy inequality (Lemma 2.5, [22, Lemma 2, p. 12]). For (1^k) , where every letter is a cut letter, (iii) gives the completeness theorem of the thesis [48, Theorem 3.5, printed pp. 19–20, PDF pp. 23–24].

Lemma 3.2 (A minor at a middle letter). *Let w have content $\mathbf{n}$, let $1 \leq j < m < l \leq k$, put $G = |w|_{j..m-1}$ and $Y = |w|_{m+1..l}$, and for $1 \leq t \leq n_m$ let G_t (Y_t) be the number of occurrences of $j \cdots (m-1)$ that lie before (of $(m+1) \cdots l$ that lie after) the t -th letter m of w .*

- (i) $|w|_{j..m} = \sum_t G_t$, $|w|_{m..l} = \sum_t Y_t$, $|w|_{j..l} = \sum_t G_t Y_t$, $G_1 \leq \cdots \leq G_{n_m} \leq G$ and $Y \geq Y_1 \geq \cdots \geq Y_{n_m} \geq 0$.
- (ii) $M_{j,m,l}(w) := |w|_{j..l} - Y|w|_{j..m} - G|w|_{m..l} + n_m GY = \sum_{t=1}^{n_m} (G - G_t)(Y - Y_t) \geq 0$.
- (iii) $M_{j,m,l}(w)$ is the minor of $\Psi(w)$ on the rows $j, m, m+1$ and the columns $m, m+1, l+1$.
- (iv) $M_{j,m,l}(w) > 0$ if and only if some occurrence of $(m+1)m(m-1)$ in w has its letter $m+1$ starting an occurrence of $(m+1) \cdots l$ and its letter $m-1$ ending an occurrence of $j \cdots (m-1)$. In particular, $M_{j,m,l}(w) = 0$ if w avoids $(m+1)m(m-1)$.
- (v) If $(j, l) = (m-1, m+1)$, then $M_{j,m,l}(w) = |w|_{(m+1)m(m-1)}$.
- (vi) If $j < m-1$ or $l > m+1$, the decreasing word $k^{n_k} \cdots 2^{n_2} 1^{n_1}$ contains $(m+1)m(m-1)$ and has $M_{j,m,l} = 0$; so the vanishing in (iv) has no converse.

Proof. (i) An occurrence of $j \cdots l$ is the t -th letter m , for some t , with an occurrence of $j \cdots (m-1)$ before it and one of $(m+1) \cdots l$ after it, and conversely; this gives the three sums. An occurrence of $j \cdots (m-1)$ before the t -th m is before the next one, so G_t is nondecreasing; likewise Y_t is nonincreasing. (ii) Expand the sum and use (i); every factor is nonnegative by (i). (iii) Entry (r, s) of $\Psi(w)$ is $|w|_{r..s-1}$, so the rows of the submatrix are $(G, |w|_{j..m}, |w|_{j..l})$, $(1, n_m, |w|_{m..l})$, $(0, 1, Y)$; expanding along the first row gives $G(n_m Y - |w|_{m..l}) - Y|w|_{j..m} + |w|_{j..l}$. (iv) By (ii), the minor is positive if and only if $G_t < G$ and $Y_t < Y$ for some t , that is, some occurrence of $j \cdots (m-1)$ ends after the t -th m and some occurrence of $(m+1) \cdots l$ starts before it. The first letter of the latter, the t -th m and the last letter of the former form an occurrence of $(m+1)m(m-1)$ of the stated kind, and every such occurrence arises in this way from its middle letter. (v) Here $G = n_{m-1}$ and $Y = n_{m+1}$, so $G - G_t$ and $Y - Y_t$ are the

numbers of letters $m - 1$ after and of letters $m + 1$ before the t -th m ; the sum in (ii) counts the occurrences of $(m + 1)m(m - 1)$ by their middle letter. (vi) The decreasing word contains $(m + 1)m(m - 1)$, as $\mathbf{n}$ is positive. In it every letter precedes every smaller one, so $|w|_{p..q} = 0$ for $p < q$ and the minor is $n_m GY$, where $Y = 0$ if $l > m + 1$ and $G = 0$ if $j < m - 1$. $\square$

Part (i) is the representation of [22, §7, pp. 15–16] with the middle word m , and the identity in (ii) is a direct consequence of it (also [19, §5, p. 8]). The inequality (ii), in the minor form (iii), is the fourth display after [29, Theorem 6, p. 196] for the words $j \cdots (m - 1)$, m , $(m + 1) \cdots l$; every minor of a Parikh matrix is nonnegative [22, Theorem 6, p. 12], [4, Theorem 4, p. 8]. Part (v) is the identity (2.2), $z - \lambda = |w|_{321}$ [22, Theorem 3, p. 7], [49, eq. (2.8), p. 3], for the subword on the letters $m - 1, m, m + 1$, renamed 1, 2, 3: for $w = 432312$, $M_{2,3,4}(w) = |w|_{432} = 3$, while $|w|_{321} = 1$. Parts (iv) and (vi) were not located (on strictness, “a general theory is missing” [29, p. 196]; limits [30, 33], Section 1).

Call two edges i and $i + 1$ *linked* if neither is extreme and the letter $i + 1$ is not a cut letter.

Corollary 3.3. *If no two edges of $\mathbf{x}$ are linked, then $\psi_2^{-1}(\mathbf{x})$ is a single Parikh class. In particular, if $\mathbf{x}$ agrees with the vector of an alternating class outside a set S of pairwise non-adjacent edges, then $\psi_2^{-1}(\mathbf{x})$ is a single Parikh class: the union of the trace classes whose projections on the edges of S have the prescribed areas.*

Proof. By induction on $l - j$, $|w|_{j..l}$ is constant on $\psi_2^{-1}(\mathbf{x})$: it is x_j if $l - j = 1$; if $l - j \geq 2$, the edges j and $j + 1$ are not linked, so one of them is extreme or $j + 1$ is a cut letter, and a part of Lemma 3.1 writes $|w|_{j..l}$ as 0 or as a product of content numbers and entries over shorter intervals. In the second statement only edges of S can be non-extreme, so none are linked, and extreme edges have single projections; so the fibre is the stated union. $\square$

Corollary 3.4 (Local vanishing rules). *Let w have content $\mathbf{n}$ and $\psi_2(w) = \mathbf{x}$, let $2 \leq m \leq k - 1$, put $(a, b, c) = (n_{m-1}, n_m, n_{m+1})$, and let $j \leq m - 1$ and $l \geq m + 1$.*

- (i) *If $x_{m-1} + x_m \leq b$, then $|w|_{(m-1)m(m+1)} = 0$, and hence $|w|_{j..l} = 0$.*
- (ii) *If $(ab - x_{m-1}) + (bc - x_m) \leq b$, then $|w|_{(m+1)m(m-1)} = 0$, and hence, with G and Y as in Lemma 3.2,*

$$|w|_{j..l} = Y|w|_{j..m} + G|w|_{m..l} - bGY. \quad (3.1)$$

Proof. Let A_t (C_t) be the number of letters $m - 1$ before (letters $m + 1$ after) the t -th letter m . By Lemma 3.2(i) for $(m - 1, m, m + 1)$, A is nondecreasing, C is nonincreasing, $\sum_t A_t = x_{m-1}$, $\sum_t C_t = x_m$ and $|w|_{(m-1)m(m+1)} = \sum_t A_t C_t$. (i) If $A_t C_t > 0$, then $A_s \geq 1$ for $s \geq t$ and $C_s \geq 1$ for $s \leq t$, so $x_{m-1} + x_m \geq (b - t + 1) + t = b + 1$. Hence every $A_t C_t$ vanishes; and every occurrence of $j \cdots l$ contains one of $(m - 1)m(m + 1)$. (ii) By Lemma 3.2(v), $|w|_{(m+1)m(m-1)} = \sum_t (a - A_t)(c - C_t)$. If the t -th term is positive, then $a - A_s \geq 1$ for $s \leq t$ and $c - C_s \geq 1$ for $s \geq t$, so $(ab - x_{m-1}) + (bc - x_m) \geq t + (b - t + 1) = b + 1$. Hence w avoids $(m + 1)m(m - 1)$, and Lemma 3.2(iv) gives $M_{j,m,l}(w) = 0$, which is (3.1). $\square$

Definition 3.5 (Rigid cells and vectors). Let (a, b, c) be a ternary content. A *cell* is a pair (x, y) with $0 \leq x \leq ab$, $0 \leq y \leq bc$, and it is *rigid* if it satisfies one of the rules

$$\begin{array}{ll} \text{(E)} & x \in \{0, ab\} \text{ or } y \in \{0, bc\}, \\ \text{(Ch)} & (ab - x) + (bc - y) \leq b, \end{array} \quad \begin{array}{ll} \text{(N)} & x + y \leq b, \\ \text{(I)} & x \in \gamma(a, b) \text{ and } y \in \gamma(b, c), \end{array}$$

with γ as in Lemma 2.6. For a content $\mathbf{n}$ with $k \geq 3$, a vector $\mathbf{x}$ in the box is a *rigid vector* if every *triple* $i \in [k - 2]$ is rigid, that is, if (x_i, x_{i+1}) is a rigid cell of (n_i, n_{i+1}, n_{i+2}) .

The rules name an extreme edge, a null triple count, the chamber of [49] and the γ -property; at $(j, l) = (m - 1, m + 1)$, Corollary 3.4(ii) is [49, Theorem 2, p. 5] and Corollary 3.4(i) its image under reversal (2.2), which exchanges (N) and (Ch) and preserves (E) and (I).

Theorem A (Rigidity). *Let $\mathbf{n}$ be a positive content with $k \geq 3$ letters.*

- (i) *Let $k = 3$ and $\mathbf{n} = (a, b, c)$. The triple count $z = |w|_{123}$ is constant on the fibre of a cell (x, y) if and only if the cell is rigid. The value is then 0 if $x = 0, y = 0$ or (N) holds; ay if $x = ab$; cx if $y = bc$; λ under (Ch); and $A \cdot C$, for the unique pair of profiles, under (Γ) .*
- (ii) *For every $\mathbf{x}$ in the box, $\psi_2^{-1}(\mathbf{x})$ is a single Parikh class if and only if $\mathbf{x}$ is a rigid vector.*
- (iii) *ψ_2 determines the Parikh matrix on $\mathcal{W}(\mathbf{n})$ if and only if $\min(n_i, n_{i+1}) = 1$ for every i .*
- (iv) *For $1 \leq i \leq k-2$ let $\mathbf{T}_i$ be the 0/1 matrix indexed by $\{0, \dots, n_i n_{i+1}\} \times \{0, \dots, n_{i+1} n_{i+2}\}$ whose (u, v) entry is 1 if and only if (u, v) is a rigid cell of (n_i, n_{i+1}, n_{i+2}) . The number $\text{Rig}(\mathbf{n})$ of rigid vectors, which by (ii) is the number of ψ_2 -fibres of content $\mathbf{n}$ that are single Parikh classes, is $\mathbf{1}^\top \mathbf{T}_1 \cdots \mathbf{T}_{k-2} \mathbf{1}$. For $\text{Rig}_k = \text{Rig}(2^k)$, with $\text{Rig}_1 = 1$ and $\text{Rig}_2 = 5$ counting all vectors,*

$$\sum_{k \geq 1} \text{Rig}_k t^k = \frac{t(1-t)(1+2t)}{1-4t-2t^2+4t^3}; \quad (3.2)$$

so $\text{Rig}_k = 4 \text{Rig}_{k-1} + 2 \text{Rig}_{k-2} - 4 \text{Rig}_{k-3}$ for $k \geq 4$, the values for $k = 3, 4, 5, 6$ are 20, 86, 364, 1548, and Rig_k / θ^k tends to a positive constant, where $4.249 < \theta < 4.25$ is the largest root of $t^3 - 4t^2 - 2t + 4$. For every sequence $n_1, n_2, \dots$ of positive integers of period p , the series $\sum_{k \geq 3} \text{Rig}(n_1, \dots, n_k) t^k$ is rational.

We call the contents of (iii), with $\min(n_i, n_{i+1}) = 1$ for every i , *isolated*. For $k = 3$, (ii) is (i), as a ternary Parikh matrix consists of the content and x, y, z . The converse of (i) is proved in Section 4, with Theorem C, from the three cases of Proposition 4.6.

Credits. The sufficiency in (i) and (ii) assembles known or immediate pieces (Section 1; words with equal projections on every edge share their matrix [4, p. 7]). Of bounds on an entry obtained from minors, Ding and Salomaa remark that “in many instances the inequalities suffice to determine” it [4, p. 8]. The remarks that without the γ -property different Parikh matrices can have the same second and third diagonals [31, p. 228], and that their fourth diagonals “can be entirely different” [4, p. 14], concern two matrices (as does [34, p. 106]); for a given vector they can fail, as for the content $(2, 3, 2)$, where $\gamma(2, 3) = \gamma(3, 2) = \{0, 1, 5, 6\}$ but $z = 0$ on the whole fibre of the cell $(2, 1)$, by (N), and $z = 6$ on that of $(4, 5)$, by (Ch). The converse of (i) was not located beyond the layer of index 1 with $a, c, x', y' \geq 2$, where z takes exactly the values λ and $\lambda + 1$ [49, Theorem 6, eq. (5.1), p. 11] (Remark 4.12), and the cells $(i^2 - i, i)$ of (i, i, i) , $i \geq 2$, which take at least two values [19, §7, Theorem 8, pp. 11–12]. The necessity in (ii), which makes the condition local, was not located; a local-to-global principle of another kind, for the M-unambiguity of a word, is [37, Theorem 3.2, Corollary 3.2]. In (iii) the sufficiency is a case of the γ -property and the necessity was not located (the independence theorem [31, Theorem 4, p. 228] concerns some words of some content); nor were the counts of (iv) and (3.2), and the OEIS has no entry for 20, 86, 364, 1548 (October 2026). Limits: [30, 33] (Section 1). By (i), the largest region where the triple count is forced, left open by the remark after [49, Theorem 2] (the chamber “is not claimed to be the largest such region”, p. 6), is the set of rigid cells (Corollary 3.6, Figure 2(a)).

Proof of Theorem A. Sufficiency in (i) and (ii). Let $\mathbf{x}$ be rigid. By induction on $l - j$, $|w|_{j..l}$ is constant on $\psi_2^{-1}(\mathbf{x})$; for $l - j = 1$ it is x_j . Let $l - j \geq 2$; for $j < m < l$ the triple $m - 1$, with edges $m - 1, m \in \{j, \dots, l - 1\}$, is rigid. If one of these triples satisfies (E), one of its edges i is extreme, and $|w|_{j..l}$ is 0 or $|w|_{j..i}|w|_{i+1..l}$ by Lemma 3.1; if one satisfies (N), then $|w|_{j..l} = 0$ by Corollary 3.4(i); if one satisfies (Ch), (3.1) gives $|w|_{j..l}$ from n_m and the entries over $\{j, \dots, m - 1\}$, $\{m + 1, \dots, l\}$, $\{j, \dots, m\}$ and $\{m, \dots, l\}$. These are content numbers

or entries over shorter intervals, constant by induction. Otherwise all these triples satisfy (Γ) ; every edge i with $j \leq i < l$ belongs to one of them, so $x_i \in \gamma(n_i, n_{i+1})$, $\mathcal{A}_i(x_i)$ is a singleton (Lemma 2.6), all words of $\psi_2^{-1}(\mathbf{x})$ have the same projection on edge i , and (2.1) applies. For $k = 3$ and $(j, l) = (1, 3)$ this is the sufficiency in (i), with the values $|w|_{1..1}|w|_{23} = ay$ if $x = ab$, $|w|_{12}|w|_{3..3} = cx$ if $y = bc$; 0 if $x = 0$, $y = 0$ or under (N); $cx + ay - abc = \lambda$ under (Ch), by (3.1) with $G = a$, $Y = c$; and $A \cdot C$ under (Γ) , by Lemma 2.4.

Necessity in (ii). Suppose triple i is not rigid. By the converse of (i), proved in Section 4 without (ii)–(iv), two ternary words of content (n_i, n_{i+1}, n_{i+2}) and pair counts (x_i, x_{i+1}) have different triple counts, $\mathbf{1}^\top \mathbf{B}_1 \mathbf{B}_2 \mathbf{1}$ by (2.1). By Lemma 2.2 some $w, w' \in \psi_2^{-1}(\mathbf{x})$ have the projections of these two words on the edges $i, i+1$ and the same projections on the other edges; then $|w|_{i..i+2} = \mathbf{1}^\top \mathbf{B}_i \mathbf{B}_{i+1} \mathbf{1}$ differs on w and w' .

(iii). If $\min(n_i, n_{i+1}) = 1$ for every i , every $\gamma(n_i, n_{i+1})$ is the whole range, every triple satisfies (Γ) , and (ii) applies. Conversely, let $\min(n_i, n_{i+1}) \geq 2$. The mirror map (Lemma 2.3) reverses the content and maps fibres onto fibres and Parikh classes onto Parikh classes, so we may assume $i \leq k-2$. Put $(a, b, c) = (n_i, n_{i+1}, n_{i+2})$, so $a, b \geq 2$, and take $\mathbf{x}$ in the box with $x_i = 2$ and $x_{i+1} = b-1$. Triple i is not rigid: (Γ) and (E) fail as $2 \notin \gamma(a, b) = \{0, 1, ab-1, ab\}$ ($ab \geq 4$) and $1 \leq b-1 < bc$; (N) fails as $x_i + x_{i+1} = b+1$; and (Ch) fails as $(ab-2) + (bc-b+1) \geq 2b-1 > b$. By (ii), $\psi_2^{-1}(\mathbf{x})$ is not a single class.

(iv). Rigidity of triple i depends only on (x_i, x_{i+1}) , so $\mathbf{x}$ is rigid if and only if $\prod_i (\mathbf{T}_i)_{x_i, x_{i+1}} = 1$; sum over the box. For $(2, 2, 2)$, $\gamma(2, 2) = \{0, 1, 3, 4\}$, and (u, v) is not rigid if and only if $u, v \in \{1, 2, 3\}$, $2 \in \{u, v\}$ and $3 \leq u+v \leq 5$: exactly the five cells $(1, 2)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 2)$ (Figure 2(b)). This set is invariant under $(u, v) \mapsto (4-u, 4-v)$, so $\mathbf{T} = \mathbf{T}_i$ maps the span of $\mathbf{f}_0 = \mathbf{e}_0 + \mathbf{e}_4$, $\mathbf{f}_1 = \mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{f}_2 = \mathbf{e}_2$, which contains $\mathbf{1} = \mathbf{f}_0 + \mathbf{f}_1 + \mathbf{f}_2$, to itself: $\mathbf{T}\mathbf{f}_0 = 2\mathbf{f}_0 + 2\mathbf{f}_1 + 2\mathbf{f}_2$, $\mathbf{T}\mathbf{f}_1 = 2\mathbf{f}_0 + 2\mathbf{f}_1$, $\mathbf{T}\mathbf{f}_2 = \mathbf{f}_0$. As $\mathbf{1}^\top \mathbf{f}_0 = \mathbf{1}^\top \mathbf{f}_1 = 2$ and $\mathbf{1}^\top \mathbf{f}_2 = 1$, $\text{Rig}_k = (2, 2, 1) \mathbf{M}^{k-2} (1, 1, 1)^\top$ for $k \geq 2$, with $\mathbf{M} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$; as $\det \mathbf{M} = -4$ and $\mathbf{M}^{-1} (1, 1, 1)^\top = (\frac{1}{2}, 0, 0)^\top$, the formula also gives $\text{Rig}_1 = 1$ at $k = 1$. Since $\det(tI - \mathbf{M}) = t^3 - 4t^2 - 2t + 4$, the Cayley–Hamilton theorem gives the recurrence for $k \geq 4$. With $\text{Rig}_1 = 1$, $\text{Rig}_2 = 5$ and $\text{Rig}_3 = (2, 2, 1) \mathbf{M} (1, 1, 1)^\top = (2, 2, 1) \cdot (5, 4, 2) = 20$ it gives 86, 364, 1548 and (3.2), whose numerator is $\text{Rig}_1 t + (\text{Rig}_2 - 4 \text{Rig}_1) t^2 + (\text{Rig}_3 - 4 \text{Rig}_2 - 2 \text{Rig}_1) t^3 = t + t^2 - 2t^3$. The cubic changes sign on $(-2, -1)$, $(0, 1)$ and $(4.249, 4.25)$, so its roots are real and θ lies in the last interval. As $\mathbf{M}^2$ is positive, the Perron–Frobenius theorem makes θ a simple eigenvalue, larger in modulus than the others, with positive eigenvectors $\mathbf{M}\mathbf{r} = \theta\mathbf{r}$, $\mathbf{1}^\top \mathbf{M} = \theta \mathbf{1}^\top$, $\mathbf{1}^\top \mathbf{r} = 1$; so $\text{Rig}_k / \theta^k \rightarrow \theta^{-2} ((2, 2, 1)\mathbf{r}) (\mathbf{1}^\top (1, 1, 1)^\top) > 0$. Finally, if $n_{i+p} = n_i$, then $\mathbf{T}_{i+p} = \mathbf{T}_i$, and for $k-2 = pq + s$ with $0 \leq s < p$ the count is $\mathbf{1}^\top \Pi^q \mathbf{T}_1 \cdots \mathbf{T}_s \mathbf{1}$ with $\Pi = \mathbf{T}_1 \cdots \mathbf{T}_p$; by the Cayley–Hamilton theorem for Π , each residue class of k modulo p gives a sequence with a linear recurrence, so the generating function is rational. $\square$

Corollary 3.6. *Let $a, b, c \geq 2$. The rigid cells of (a, b, c) are those on the boundary of the box, by (E); those with $x+y \leq b$, by (N), where $z = 0$; those with $x'+y' \leq b$, by (Ch), where $z = \lambda$; and the cells $(1, bc-1)$ and $(ab-1, 1)$ of (Γ) alone, where $z = c-1$ and $z = a-1$.*

Proof. Here $\gamma(a, b) = \{0, 1, ab-1, ab\}$ and $\gamma(b, c) = \{0, 1, bc-1, bc\}$. A (Γ) cell with a coordinate in $\{0, ab\}$ or $\{0, bc\}$ satisfies (E); of the other four, $(1, 1)$ satisfies (N) and $(ab-1, bc-1)$ satisfies (Ch) as $b \geq 2$, while $(1, bc-1)$ and $(ab-1, 1)$ satisfy neither, as $bc > b$ and $ab > b$. Their profiles are $A = (0, \dots, 0, 1)$, $C = (c, \dots, c, c-1)$ and $A = (a-1, a, \dots, a)$, $C = (1, 0, \dots, 0)$, so $z = A \cdot C$ is $c-1$ and $a-1$. $\square$

Corollary 3.7 (The fourth diagonal decides). *Let $\mathbf{n}$ be a positive content with $k \geq 3$ and $\mathbf{x}$ a vector in the box. Then $\psi_2^{-1}(\mathbf{x})$ is a single Parikh class if and only if every entry $|w|_{i(i+1)(i+2)}$ of the fourth diagonal of Ψ is constant on $\psi_2^{-1}(\mathbf{x})$.*

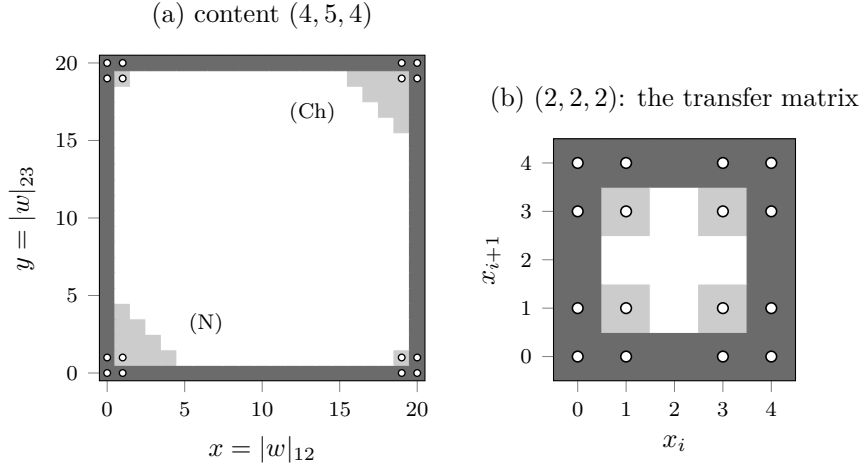


Figure 2: (a) Content (4, 5, 4): the 102 rigid cells of the 441 cells (x, y) of $[0, 20]^2$, where $z = |w|_{123}$ takes one value on the fibre (Theorem A). Dark: rule (E), the edges. Light: the other rigid cells, by (N), $x + y \leq 5$, by (Ch), $(20 - x) + (20 - y) \leq 5$, or by (Γ) alone. Dots: (Γ), $x, y \in \{0, 1, 19, 20\}$. The four rules hold on 80, 21, 21 and 16 cells, with overlaps; (1, 19) and (19, 1) satisfy only (Γ). (b) The same map for (2, 2, 2) is the transfer automaton of (2^k) on the edge values $0, \dots, 4$, as its matrix $\mathbf{T}_i$, the same for every i : the fibre of a ψ_2 -vector of (2^k) is a single Parikh class if and only if every step (x_i, x_{i+1}) is shaded, so $\text{Rig}_k = \mathbf{1}^\top \mathbf{T}_1 \cdots \mathbf{T}_{k-2} \mathbf{1} = 20, 86, 364, 1548$ for $k = 3, 4, 5, 6$.

Proof. The direct implication is trivial. If $|w|_{i..i+2}$ is constant on $\psi_2^{-1}(\mathbf{x})$, then, since every pair of projections on the edges $i, i+1$ with areas x_i, x_{i+1} occurs in $\psi_2^{-1}(\mathbf{x})$ (Lemma 2.2) and determines $|w|_{i..i+2}$ by (2.1), z is constant on the fibre of the cell (x_i, x_{i+1}) of (n_i, n_{i+1}, n_{i+2}) , which is therefore rigid by Theorem A(i). Apply Theorem A(ii). $\square$

4 Ternary fibres: extremes and the top of a fibre

Off the rigid cells a ternary fibre splits into several Parikh classes, one for each value of $z = |w|_{123}$ in $\mathcal{Z}(x, y)$ (Lemma 2.4). We find the extremes of $\mathcal{Z}(x, y)$ (Theorem B) and its top (Theorem C), completing the proof of Theorem A(i); neither result describes the whole set. Throughout, (x, y) is a cell (Definition 3.5) of (a, b, c) , in the notation of Section 2, and for $1 \leq t \leq b$

$$\begin{aligned} A_t^{\text{bal}} &= q_x + [t > b - r], & C_t^{\text{bal}} &= q_y + [t \leq s], \\ A_t^{\text{con}} &= a[t > b - Q] + R[t = b - Q], & C_t^{\text{con}} &= c[t \leq T] + S[t = T + 1]. \end{aligned}$$

The *balanced* profiles spread evenly and the *concentrated* ones fill from the right and the left: as $r + t - 1 \leq 2b - 2$ and $x - a(b - t) = a(t - b + Q) + R$ (likewise for y), $A_t^{\text{bal}} = \lfloor (x + t - 1)/b \rfloor$, $C_t^{\text{bal}} = \lfloor (y + b - t)/b \rfloor$, $A_t^{\text{con}} = \min(a, (x - a(b - t))_+)$ and $C_t^{\text{con}} = \min(c, (y - c(t - 1))_+)$.

4.1 The least and greatest triple counts

Theorem B (Exact range). *Let (x, y) be a cell of the content (a, b, c) .*

- (i) $z_{\max} = bq_xq_y + q_xs + q_yr + \max(0, r + s - b)$. *Equivalently, the least value of $\Delta = |w|_{2132} + |w|_{2312}$ on $\psi_2^{-1}(x, y)$ is $\min(rs, (b - r)(b - s))$.*
- (ii) $z_{\min} = ac(Q + T - b) + aS + cR = \lambda$ if $Q + T \geq b$; $z_{\min} = RS$ if $Q + T = b - 1$; and $z_{\min} = 0$ if $Q + T \leq b - 2$. *Equivalently, the least value of $|w|_{321} = z - \lambda$ on $\psi_2^{-1}(x, y)$ is $\max(0, -\lambda) + [Q + T = b - 1] \min(RS, (a - R)(c - S))$.*

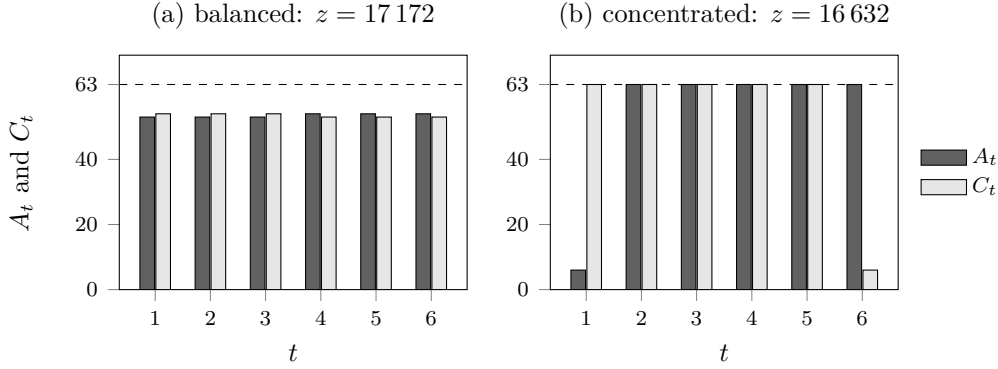


Figure 3: Profile pairs of the cell $(x, y) = (321, 321)$ of content $(a, b, c) = (63, 6, 63)$, the cell of the word α_5 of Ding and Salomaa [4]. For the t -th letter 2 of a word, A_t (dark) is the number of 1s before it and C_t (light) the number of 3s after it, so that $x = \sum_t A_t$, $y = \sum_t C_t$ and $z = |w|_{123} = \sum_t A_t C_t$; the dashed line is $a = c = 63$. (a) The balanced pair $A^{\text{bal}} = (53, 53, 53, 54, 54, 54)$, $C^{\text{bal}} = (54, 54, 54, 53, 53, 53)$ gives $z_{\max} = 17\,172$, one less than $\lfloor xy/b \rfloor = 17\,173$. (b) The concentrated pair $A^{\text{con}} = (6, 63, 63, 63, 63, 63)$, $C^{\text{con}} = (63, 63, 63, 63, 63, 6)$ gives $z_{\min} = 16\,632 = \lambda = cx + ay - abc$ (Theorem B).

Moreover $z_{\max} = A^{\text{bal}} \cdot C^{\text{bal}}$ and $z_{\min} = A^{\text{con}} \cdot C^{\text{con}}$.

Theorem B gives the least remainders of the bounds of Lemma 2.5 on every fibre. Part (i) for $b \mid x$ or $b \mid y$ is a direct consequence of the equality case of [22, Lemma 2] (Corollary 4.2); the branch $Q + T \geq b$ of (ii) is [49, Lemma 4, eq. (4.1), p. 8], whose criterion $\lceil x'/a \rceil + \lceil y'/c \rceil \leq b$ is noted there to be $Q + T \geq b$ (Corollary 4.4); and [19, §§4 and 7] prints instances. The other cases were not located (limits: [33, 30], unread, and [44, 45, 13], abstracts or metadata only).

Lemma 4.1. $A^{\text{bal}}, A^{\text{con}} \in \mathcal{A}(x)$, $C^{\text{bal}}, C^{\text{con}} \in \mathcal{C}(y)$; for $A \in \mathcal{A}(x)$, $C \in \mathcal{C}(y)$ and $1 \leq \ell < b$,

$$\max(0, x - a(b - \ell)) = A_{\leq \ell}^{\text{con}} \leq A_{\leq \ell} \leq A_{\leq \ell}^{\text{bal}} = \ell q_x + \max(0, \ell - b + r), \quad (4.1)$$

$$\max(0, y - c\ell) = C_{> \ell}^{\text{con}} \leq C_{> \ell} \leq C_{> \ell}^{\text{bal}} = (b - \ell)q_y + \max(0, s - \ell). \quad (4.2)$$

Proof. The profiles are monotone with sums $bq_x + r$, $bq_y + s$, $aQ + R$, $cT + S$. If $r \geq 1$ then $bq_x < x \leq ab$, so $q_x + 1 \leq a$ (likewise for s); if $R \geq 1$ then $aQ < x \leq ab$, so the position $b - Q \geq 1$ exists (likewise for S). In (4.1), $A_{\leq \ell} \geq 0$, and $A_{\leq \ell} \geq x - a(b - \ell)$ as the last $b - \ell$ entries are at most a . If $A_\ell \leq q_x$ then $A_{\leq \ell} \leq \ell q_x$; otherwise $A_t \geq q_x + 1$ for $t \geq \ell$, and $A_{\leq \ell} \leq x - (b - \ell)(q_x + 1) = \ell q_x + \ell - b + r$. As the entries $q_x + 1$ of A^{bal} fill its last r positions, $A_{\leq \ell}^{\text{bal}}$ is as stated. If $b - \ell \leq Q$, the last $b - \ell$ entries of A^{con} equal a and $A_{\leq \ell}^{\text{con}} = x - a(b - \ell) \geq 0$; otherwise they contain all nonzero entries, and $A_{\leq \ell}^{\text{con}} = 0 > R - a \geq x - a(b - \ell)$. Finally, (4.2) is (4.1) for the nondecreasing vector $(C_b, \dots, C_1)$ and $(c, y, b - \ell)$ in place of (a, x, ℓ) , whose balanced and concentrated profiles are the reversals of C^{bal} and C^{con} . $\square$

Proof of Theorem B. For $A, C \in \mathbb{R}^b$, writing $C_t = C_b + \sum_{\ell \geq t} (C_\ell - C_{\ell+1})$, or $A_t = A_1 + \sum_{\ell < t} (A_{\ell+1} - A_\ell)$, and exchanging the sums gives the summation by parts

$$A \cdot C = C_b \sum_t A_t + \sum_{\ell=1}^{b-1} (C_\ell - C_{\ell+1}) A_{\leq \ell} = A_1 \sum_t C_t + \sum_{\ell=1}^{b-1} (A_{\ell+1} - A_\ell) C_{> \ell}. \quad (4.3)$$

Let $A \in \mathcal{A}(x)$, $C \in \mathcal{C}(y)$. In the left-hand form the coefficients $C_\ell - C_{\ell+1}$ are nonnegative and $x C_b$ does not depend on A , so (4.1) gives $A^{\text{con}} \cdot C \leq A \cdot C \leq A^{\text{bal}} \cdot C$; likewise the right-hand form, for nondecreasing A , and (4.2) give $A \cdot C^{\text{con}} \leq A \cdot C \leq A \cdot C^{\text{bal}}$. Hence

$A^{\text{con}} \cdot C^{\text{con}} \leq A^{\text{con}} \cdot C \leq A \cdot C \leq A^{\text{bal}} \cdot C \leq A^{\text{bal}} \cdot C^{\text{bal}}$, and the outer pairs, realised by words (Lemmas 4.1 and 2.4), give $z_{\min}$ and $z_{\max}$.

Expanding, $A^{\text{bal}} \cdot C^{\text{bal}} = bq_xq_y + q_xs + q_yr + \#\{t : b - r < t \leq s\} = z_{\max}$, and

$$A^{\text{con}} \cdot C^{\text{con}} = ac \cdot \#\{t : b - Q < t \leq T\} + aS[b - Q < T + 1 \leq b] \\ + cR[1 \leq b - Q \leq T] + RS[b - Q = T + 1].$$

The count is $\max(0, Q + T - b)$; the second and third brackets are $[Q + T \geq b]$, as $T + 1 \leq b$ fails only if $T = b$, when $S = 0$, and $b - Q \geq 1$ only if $Q = b$, when $R = 0$; the last is $[Q + T = b - 1]$. As $ac(Q + T - b) + aS + cR = c(aQ + R) + a(cT + S) - abc = \lambda$, this is $z_{\min}$.

For the equivalent forms, $xy - bz_{\max} = rs - b \max(0, r + s - b)$ is rs if $r + s \leq b$ and $(b - r)(b - s)$ otherwise, the smaller of the two; $\Delta = |w|_{2132} + |w|_{2312}$ by Lemma 2.5. By (2.2), $|w|_{321} = z - \lambda$ has least value $z_{\min} - \lambda$. If $Q + T \geq b$ this is 0, and $\lambda \geq 0$. If $Q + T \leq b - 2$ it is $-\lambda$, and $\lambda \leq -2ac + a(c - 1) + c(a - 1) = -(a + c)$. If $Q + T = b - 1$, then $\lambda = RS - (a - R)(c - S)$ and $z_{\min} - \lambda = (a - R)(c - S) = \max(0, -\lambda) + \min(RS, (a - R)(c - S))$ for either sign of λ . $\square$

Corollary 4.2. *The bound xy/b is attained on $\psi_2^{-1}(x, y)$ if and only if b divides x or y .*

Proof. By Theorem B(i) this happens if and only if $rs = 0$, as $(b - r)(b - s) \geq 1$; equivalently, by the equality case in Lemma 2.5, A or C is constant, which is possible exactly if b divides x or y . $\square$

Corollary 4.3. $\lfloor xy/b \rfloor - z_{\max} = \lfloor \min(rs, (b - r)(b - s))/b \rfloor \leq \lfloor b/4 \rfloor$; for $b \geq 2$, equality holds at $(x, y) = (\lfloor b/2 \rfloor, \lceil b/2 \rceil)$, where $z_{\max} = 0$.

Proof. Let $d = \min(rs, (b - r)(b - s))$; then $xy = bz_{\max} + d$, so $\lfloor xy/b \rfloor = z_{\max} + \lfloor d/b \rfloor$. By AM–GM, applied to rs if $r + s \leq b$ and to $(b - r)(b - s)$ otherwise, $d \leq b^2/4$, so $\lfloor d/b \rfloor \leq \lfloor b/4 \rfloor$. At $(\lfloor b/2 \rfloor, \lceil b/2 \rceil)$, $q_x = q_y = 0$ and $r + s = b$, so $z_{\max} = 0$ and $d = \lfloor b^2/4 \rfloor$; then $\lfloor d/b \rfloor = \lfloor b/4 \rfloor$, as d/b is $b/4$ if $4 \mid b$ and otherwise lies between $(b^2 - 1)/(4b) \geq (b - 1)/4 \geq \lfloor b/4 \rfloor$ and $b/4$. $\square$

Corollary 4.4. (i) $z_{\min} \geq \lambda$, with equality if and only if $Q + T \geq b$; equivalently, some word of the fibre avoids 321 if and only if $Q + T \geq b$.

(ii) If $Q + T = b - 1$, $z_{\min} - \lambda = (a - R)(c - S) \geq 1$; if $Q + T \leq b - 2$, $\lambda \leq -(a + c) < 0 = z_{\min}$. So $\max(0, \lambda)$ is attained if and only if $Q + T \neq b - 1$ or $RS = 0$.

Proof. See the proof of Theorem B(ii): $z_{\min} = \lambda + \min|w|_{321}$, and $(a - R)(c - S) \geq 1$. So the bound $\lambda > 0$ of [4, Lemma 3] is not attained if $Q + T = b - 1$ and $RS > (a - R)(c - S)$. $\square$

Example 4.5. Ding and Salomaa tabulate their bounds λ and $\lfloor xy/b \rfloor$, valid and not claimed to be attained, for the words $\alpha_i = \prod_{j=0}^i 1^{2^i-j} 2^{2^j}$, $i \leq 5$ (in our letters; [4, pp. 11–12]). The word α_5 has content (63, 6, 63), pair counts (321, 321) and printed bounds 16 632 and 17 173. As $Q = T = 5$, $z_{\min} = \lambda = 16\,632$ is attained; but $r = s = 3$, so the least Δ is 9 and $z_{\max} = 17\,173 - \lfloor 9/6 \rfloor = 17\,172$ (Figure 3). Every integer from 16 632 to 17 172 occurs on this fibre (checks/check_r2_extrema.py, section [8], gives a witness word for each).

4.2 The top of a fibre

Values between $z_{\min}$ and $z_{\max}$ may be missing: for $(2, 2, 2)$ the fibre of $(2, 2)$ has only the values 0 and 2, on 6 words each. In the example of [22, §8, pp. 19–20] (also [19, §4, pp. 6–7]), content (3, 5, 3), all values from 0 to 8 occur at $(7, 6)$, while at $(7, 7)$ the nonnegativity of the minors [22, Theorem 6, p. 12] allows $0 \leq z \leq 9$ and the values are 1, 2, 4, 5, 6, 7, 8, 9: Theorem B gives $z_{\max} = 9$ ($r = s = 2$) and $z_{\min} = RS = 1$ ($Q + T = b - 1$), and 3 is a hole. Just below the top, holes are rare and explicit.

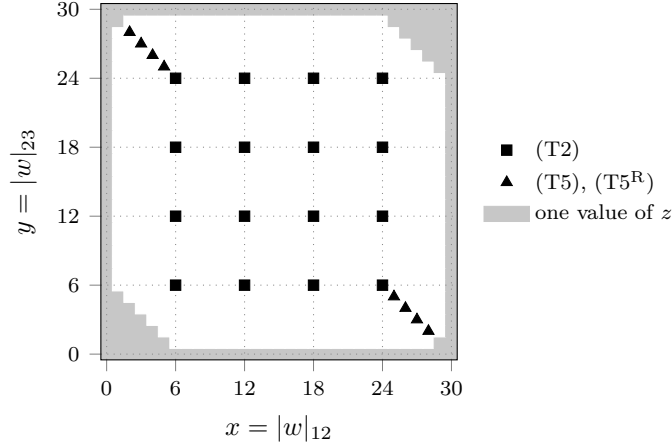


Figure 4: The top-gap map of content $(5, 6, 5)$ on the box $[0, 30]^2$. Squares: the 16 cells (T2), $(x, y) = (6u, 6v)$ with $1 \leq u, v \leq 4$. Triangles: the 8 cells (T5), $(x, 30 - x)$ with $2 \leq x \leq 5$, and (T5^R), $(30 - y, y)$ with $2 \leq y \leq 5$. These 24 cells are exactly those where $z = |w|_{123}$ takes at least two values on the fibre and $z_{\max} - 1$ does not occur; at each of them $z_{\max} - 2$ occurs (Theorem C). Grey: the rigid cells, where z takes one value. At every other cell $z_{\max} - 1$ occurs.

Theorem C (The top of a fibre). *Let $b \geq 2$ and let (x, y) be a cell with $|\mathcal{Z}(x, y)| \geq 2$. Then $z_{\max} - 1 \notin \mathcal{Z}(x, y)$ if and only if one of the following holds:*

(T2) $r = s = 0$, $1 \leq q_x \leq a - 1$ and $1 \leq q_y \leq c - 1$;

(T5) $2 \leq x \leq b - 1$ and $y = bc - x$;

(T5^R) $2 \leq y \leq b - 1$ and $x = ab - y$.

In these cases $z_{\max} - 2 \in \mathcal{Z}(x, y)$.

Under (T2), $z_{\max} = xy/b$ is the attained Cauchy bound, and Theorem C says that $\Delta \neq b$ on the fibre. The hypothesis $|\mathcal{Z}(x, y)| \geq 2$ is needed, as $z_{\max} - 1$ is also missing on every rigid fibre. Figure 4 shows the cells of Theorem C for $(5, 6, 5)$. Theorem C as a statement and the counts of Proposition 4.10 were not located (limits as for Theorem B: [33, 30, 44, 45, 13]). Two instances of (T2) are printed in [19, §7, Theorem 8, pp. 11–12], at the cell $(i^2 - i, i)$ of (i, i, i) ($b = i$, $q_x = i - 1$, $q_y = 1$): the theorem excludes the values strictly between 0 and i , which for $i = 2$ is the gap of $(2, 2, 2)$ above, and for $i = 4$ the printed list 0, 4, 6, 8, 9, 10, 12 omits $z_{\max} - 1 = 11$ and contains $z_{\max} - 2 = 10$, without remark. On the layer of index 1 the conclusion follows from [49, Theorem 6] (Remark 4.12).

Call a cell *exceptional* if $b \geq 2$ and either (T2) holds, or $a, c \geq 2$ and (T5) or (T5^R) holds. Theorem C and the converse of Theorem A(i) follow from a finer statement.

Proposition 4.6 (Three cases at the top of a fibre). *Every cell is in exactly one case.*

(i) *It is rigid. Then $|\mathcal{Z}(x, y)| = 1$.*

(ii) *It is exceptional. Then $z_{\max} - 1 \notin \mathcal{Z}(x, y)$ and $z_{\max} - 2 \in \mathcal{Z}(x, y)$.*

(iii) *Neither. Then $z_{\max} - 1 \in \mathcal{Z}(x, y)$.*

The mirror map (Lemma 2.3) sends (x, y, z) to (y, x, z) for (c, b, a) and a profile pair (A, C) to $((C_b, \dots, C_1), (A_b, \dots, A_1))$; it exchanges (a, x, r, q_x) with (c, y, s, q_y) and (T5) with (T5^R), maps balanced pairs to balanced pairs, and preserves (T2) and the four rules.

Lemma 4.7 (Six moves). *Let $b \geq 2$, $(A, C) = (A^{\text{bal}}, C^{\text{bal}})$, and $\mathbf{e}_1, \dots, \mathbf{e}_b$ the unit vectors of $\mathbb{Z}^b$. A move whose conditions hold gives a pair in $\mathcal{A}(x) \times \mathcal{C}(y)$ with dot product $z_{\max} - 1$.*

Move	Change	Conditions
M1	$A' = A - \mathbf{e}_1 + \mathbf{e}_b$	$s \geq 1, q_x \geq 1, q_x + [r \geq 1] \leq a - 1$
M2	$C' = C + \mathbf{e}_1 - \mathbf{e}_b$	$r \geq 1, q_y \geq 1, q_y + [s \geq 1] \leq c - 1$
M3	$A' = A - \mathbf{e}_1 + \mathbf{e}_{b-r}$	$r + s < b, s \geq 1, 1 \leq q_x \leq a - 1$
M3'	$A' = A - \mathbf{e}_{b-r+1} + \mathbf{e}_b$	$r + s > b, q_x + 2 \leq a$
M4	$C' = C + \mathbf{e}_1 - \mathbf{e}_s$	$r + s > b, q_y + 2 \leq c$
M5	$C' = C + \mathbf{e}_{s+1} - \mathbf{e}_b$	$r + s < b, r \geq 1, 1 \leq q_y \leq c - 1$

Proof. Mirroring sends M2, M5, M4 with their conditions to M1, M3, M3', which we check. Sums are kept; $A' = A - \mathbf{e}_i + \mathbf{e}_j$ ($i < j$) changes the dot product by $C_j - C_i$. As $r, s \leq b - 1$, $A_1 = q_x$, $A_b = q_x + [r \geq 1]$, $C_1 = q_y + [s \geq 1]$ and $C_b = q_y$. M1: lowering $A_1 = q_x \geq 1$ and raising $A_b \leq a - 1$ keeps $A \in \mathcal{A}(x)$, and $C_b - C_1 = -1$ as $s \geq 1$. M3: as $s \geq 1$ and $r + s < b$, $b - r \geq 2$ is the last position with value q_x ; lowering A_1 to $q_x - 1 \geq 0$ and raising A_{b-r} to $q_x + 1 \leq a$ keeps $A \in \mathcal{A}(x)$, and $C_{b-r} - C_1 = -1$ as $s < b - r$. M3': as $r + s > b$ and $s < b$, $r \geq 2$, and $b - r + 1 \leq b - 1$ is the leftmost position with value $q_x + 1$; lowering it to q_x and raising A_b to $q_x + 2 \leq a$ keeps $A \in \mathcal{A}(x)$, and $C_b - C_{b-r+1} = -1$ as $b - r + 1 \leq s$. $\square$

Moving letters within a fibre and reading off the change of z is a known device: each rule of the “Moving Algorithm” [37, Appendix A, ms. pp. 16–17], credited there to its reference 17, changes the number of occurrences of abc by the number of letters c between the two positions involved; see also [32, §6, Lemma 4, p. 1825, Theorem 9, p. 1826] and [12, Proposition 5, pp. 5–6]. Each move is such a pair of swaps, read on the profiles; only the case analysis below is claimed, and it was not located.

Proposition 4.8 (The case tree). *Let $b \geq 2$, and let the cell be neither exceptional nor in (E). Then the conditions of one of the moves hold, or the cell satisfies (N), (Ch) or (Γ).*

Proof. As $0 < x < ab$ and $0 < y < bc$: $q_x \leq a - 1$, $q_y \leq c - 1$, $q_x \geq 1$ or $r \geq 1$, and $q_y \geq 1$ or $s \geq 1$. If $r = s = 0$, then (T2) holds, which is excluded. If $r \geq 1$ and $s = 0$, then $q_y \geq 1$ and $C_1 = q_y \leq c - 1$, so M2 applies; symmetrically M1 applies if $r = 0 < s$. If $r, s \geq 1$ and $1 \leq q_x \leq a - 2$, then $A_b = q_x + 1 \leq a - 1$ and M1 applies; symmetrically M2 applies if $r, s \geq 1$ and $1 \leq q_y \leq c - 2$. There remain $r, s \geq 1$, $q_x \in \{0, a - 1\}$, $q_y \in \{0, c - 1\}$.

- $r + s < b$: M3 applies if $q_x \geq 1$, M5 if $q_y \geq 1$; otherwise $x + y = r + s < b$, which is (N).
- $r + s > b$: M3' applies if $q_x + 2 \leq a$, M4 if $q_y + 2 \leq c$; otherwise $q_x = a - 1$, $q_y = c - 1$ and $x' + y' = 2b - r - s < b$, which is (Ch).
- $r + s = b$: $(q_x, q_y) = (0, 0)$ gives (N) and $(a - 1, c - 1)$ gives (Ch). If $q_x = 0 \neq a - 1$ and $q_y = c - 1 \neq 0$, then $a, c \geq 2$, $x = r$, $y = bc - r$: (T5) (excluded) for $r \geq 2$, and (Γ) for $r = 1$, as $1 \in \gamma(a, b)$, $bc - 1 \in \gamma(b, c)$. The case $q_x = a - 1 \neq 0$, $q_y = 0 \neq c - 1$ gives (T5^R) or (Γ) by mirroring. If $a = 1$ or $c = 1$, then (q_x, q_y) is $(0, 0)$ or $(a - 1, c - 1)$. $\square$

Lemma 4.9. *Exceptional cells are not rigid.*

Proof. Under (T2), $b \leq x \leq ab - b$ and $b \leq y \leq bc - b$, so (E), (N), (Ch) fail, and $x \notin \gamma(a, b) = \{0, 1, ab - 1, ab\}$. Under (T5) ($a, c \geq 2$): $2 \leq x < b < ab - 1$, $bc - b < y < bc - 1$, $x + y = bc > b$ and $x' + y' = ab > b$, so no rule holds; (T5^R) mirrors this. $\square$

Proposition 4.10 (Second layers). *Let $b \geq 2$. Let $\Lambda_m \in \mathbb{Z}^b$ ($0 \leq m \leq \lfloor b/2 \rfloor$) have m leftmost entries -1 , m rightmost entries 1 and the others 0 ; for $x \leq b$ and $0 \leq m \leq \lfloor x/2 \rfloor$ let $\Upsilon_m = (0^{b-x+m}, 1^{x-2m}, 2^m)$; let $\eta_2 = 6$ and $\eta_n = 2\lfloor n/2 \rfloor + 2$ for $n \geq 3$. A number plus a vector is taken entrywise.*

- (i) Under (T2), $z_{\max} = bq_xq_y$, $z_{\max} - 1 \notin \mathcal{Z}(x, y)$, the pairs with $A \cdot C = z_{\max} - 2$ are exactly the $2\lfloor b/2 \rfloor - 1$ pairs $(q_x + \Lambda_m, q_y - \Lambda_1)$, $(q_x + \Lambda_1, q_y - \Lambda_m)$, $1 \leq m \leq \lfloor b/2 \rfloor$, and

$$\rho(x, y, z_{\max} - 2) = \eta_b \binom{q_x + c - q_y - 2}{q_x - 1} \binom{a - q_x + q_y - 2}{a - q_x - 1}.$$

- (ii) Under (T5) with $a, c \geq 2$, $z_{\max} = (c - 1)x$, $z_{\max} - 1 \notin \mathcal{Z}(x, y)$, the pairs with $A \cdot C = z_{\max} - 2$ are exactly the $(\Upsilon_m, c - \Upsilon_n)$ with $1 \leq m, n \leq \lfloor x/2 \rfloor$ and $\min(m, n) = 1$, and $\rho(x, y, z_{\max} - 2) = \eta_x \binom{a+c-4}{a-2}$.

- (iii) Under (T5^R) with $a, c \geq 2$, $z_{\max} = (a - 1)y$, $z_{\max} - 1 \notin \mathcal{Z}(x, y)$ and $\rho(x, y, z_{\max} - 2) = \eta_y \binom{a+c-4}{a-2}$.

In all three cases $z_{\max} - 2 \in \mathcal{Z}(x, y)$.

Proof. (i) Write $A = q_x + U$, $C = q_y + V$ with U nondecreasing, V nonincreasing, $\sum U_t = \sum V_t = 0$. Then $A \cdot C = bq_xq_y + U \cdot V$, and by (4.3), $\delta := bq_xq_y - A \cdot C = \sum_{\ell=1}^{b-1} (V_\ell - V_{\ell+1})(-U_{\leq \ell})$. If $U \neq 0$ then $U_1 < 0 < U_b$ and $U_{\leq \ell} \leq -1$ for $1 \leq \ell < b$ (if $U_\ell \leq 0$ then $U_{\leq \ell} \leq U_1$; if $U_\ell \geq 1$ then $U_{\leq \ell} = -\sum_{t>\ell} U_t \leq -1$); likewise $V \neq 0$ gives $V_1 \geq 1$ and $V_b \leq -1$. So $\delta = 0$ if $U = 0$ or $V = 0$ (the constant $A = q_x$ is admissible), whence $z_{\max} = bq_xq_y$, and otherwise

$$\delta \geq \sum_{\ell=1}^{b-1} (V_\ell - V_{\ell+1}) = V_1 - V_b \geq 2. \quad (4.4)$$

So $\delta \neq 1$. If $\delta = 2$, equality holds in (4.4): $V_1 = 1 = -V_b$, so $V = -\Lambda_m$ with $1 \leq m \leq \lfloor b/2 \rfloor$, and $U_{\leq \ell} = -1$ at its descents $\ell = m, \ell = b - m$. If $m = 1$, then $U_1 = -1 = -U_b$, so $U = \Lambda_j$ for some $j \geq 1$. If $m \geq 2$, then $U_m \geq 0$ (the largest of m integers with sum -1) and $U_{b-m+1} \leq 0$ (the smallest of m integers with sum 1), so U vanishes on $\{m, \dots, b - m + 1\}$; the entries before (after) this interval are ≤ 0 (≥ 0) with sum -1 (1), so $U = \Lambda_1$. Conversely $\Lambda_j \cdot \Lambda_1 = 2$, and the pairs are admissible, their entries differing from q_x or q_y by at most 1.

Each such pair has $(A_1, A_b, C_1, C_b) = (q_x - 1, q_x + 1, q_y + 1, q_y - 1)$, so the gaps 0 and b hold $(\alpha_0, \gamma_0) = (q_x - 1, c - q_y - 1)$ and $(\alpha_b, \gamma_b) = (a - q_x - 1, q_y - 1)$ letters 1 and 3: the two binomials. Inside, $q_x + \Lambda_j$ increases at the gaps j and $b - j$ (by 2 at gap j if $b = 2j$), and $q_y - \Lambda_m$ decreases at the gaps m and $b - m$. For $b = 2$ the one pair puts two letters 1 and two letters 3 into gap 1: weight 6. For $b \geq 3$, $(q_x + \Lambda_1, q_y - \Lambda_1)$ puts one 1 and one 3 into each of the gaps 1 and $b - 1$: weight 4; for $2 \leq j \leq \lfloor b/2 \rfloor$, $\{j, b - j\} \cap \{1, b - 1\} = \emptyset$, so the two pairs with index j have weight 1. The total is $4 + 2(\lfloor b/2 \rfloor - 1) = \eta_b$.

(ii) For $C \in \mathcal{C}(y)$ let $B = c - C$, nondecreasing with sum x ; then $A \cdot C = cx - A \cdot B$. As $x < b$, $A_1 = B_1 = 0$, and the supports of A and B are nested final segments. If B has entries in $\{0, 1\}$, then $B = (0^{b-x}, 1^x)$; the support of A has at most x elements, so it lies in the last x positions, and $A \cdot B = x$; likewise if A has entries in $\{0, 1\}$. Otherwise $A_b, B_b \geq 2$, and if, say, the support of B lies in that of A , then $A \cdot B - x = \sum_{B_t > 0} (A_t - 1)B_t \geq (A_b - 1)B_b \geq 2$, with equality only if $A_b = B_b = 2$. A nondecreasing vector with entries in $\{0, 1, 2\}$, sum x and m entries 2 is Υ_m , and counting from the right, $\Upsilon_m \cdot \Upsilon_n = 4m + 2(n - m) + (x - 2n) = x + 2m$ for $m \leq n$. So $A \cdot B \neq x + 1$, its least value is x , and $A \cdot B = x + 2$ exactly for (Υ_m, Υ_n) with $\min(m, n) = 1$; these pairs are admissible as $\min(a, c) \geq 2$. As $z = cx - A \cdot B$, this gives $z_{\max}$, the missing value and the pairs at $z_{\max} - 2$. For the count, $(A, C) = (\Upsilon_m, c - \Upsilon_n)$ has $\alpha_0 = \gamma_0 = 0$, $\alpha_b = a - 2$, $\gamma_b = c - 2$, a factor $\binom{a+c-4}{a-2}$. The other gaps hold the increases of Υ_m (letters 1) and of Υ_n (letters 3), and Υ_m increases at the gaps $b - x + m$ and $b - m$ (once, by 2, if $x = 2m$), both in $\{2, \dots, b - 1\}$. So $(\Upsilon_1, c - \Upsilon_1)$ has weight 6 if $x = 2$ and 4 if $x \geq 3$, and for $2 \leq n \leq \lfloor x/2 \rfloor$, $\{b - x + 1, b - 1\} \cap \{b - x + n, b - n\} = \emptyset$ (as $n \neq 1, n \neq x - 1$), so $(\Upsilon_1, c - \Upsilon_n)$ and $(\Upsilon_n, c - \Upsilon_1)$ have weight 1. The total is η_x .

- (iii) This is (ii) for the content (c, b, a) , by mirroring. $\square$

Proof of Proposition 4.6, Theorem C and the converse of Theorem A(i). The cases are exclusive by Lemma 4.9; case (i) is the sufficiency half of Theorem A(i), and case (ii) is Proposition 4.10. In case (iii), $b \geq 2$, since for $b = 1$ every cell satisfies (Γ) ($\gamma(a, 1) = \{0, \dots, a\}$, $\gamma(1, c) = \{0, \dots, c\}$); the cell is neither exceptional nor rigid, in particular not in (E), so a move applies by Proposition 4.8, and $z_{\max} - 1 \in \mathcal{Z}(x, y)$ by Lemmas 2.4 and 4.7. So every cell that is not rigid has $z_{\max}$ and $z_{\max} - 1$ or $z_{\max} - 2$ in $\mathcal{Z}(x, y)$: the converse of Theorem A(i). For Theorem C, let $|\mathcal{Z}(x, y)| \geq 2$, so the cell is not rigid. Under (T5), $c = 1$ would give $x + y = b$, (N), and $a = 1$ would give $x' + y' = (b - x) + x = b$, (Ch); so $a, c \geq 2$ and the cell is exceptional, and by mirroring so is a cell under (T5^R). So case (ii) applies under the three conditions, and case (iii) otherwise. $\square$

Corollary 4.11. *Let $b \geq 2$. The number of cells of (a, b, c) with $|\mathcal{Z}(x, y)| \geq 2$ and $z_{\max} - 1 \notin \mathcal{Z}(x, y)$ is $(a - 1)(c - 1) + 2 \max(0, b - 2)$ if $a, c \geq 2$, and 0 otherwise.*

Proof. These are the exceptional cells (Proposition 4.6), which need $a, c \geq 2$. Then (T2) gives $(a - 1)(c - 1)$ cells and (T5), (T5^R) give $\max(0, b - 2)$ each, disjointly, as $b \leq x \leq ab - b$, $x < b$ and $x > ab - b$ respectively. $\square$

For the content $(5, 6, 5)$ these are the 24 cells of Figure 4.

Remark 4.12. Let $a, c \geq 2$, and let (x, y) lie on the layer of index 1, $x' + y' = b + 1$, with $x', y' \geq 2$. There [49, Theorem 6, eq. (5.1), p. 11] gives the whole fibre, on which z takes exactly the values λ and $\lambda + 1$ (at $(4, 4)$ of $(2, 3, 2)$, $z = 4$ on 7 words and $z = 5$ on 1 word [49, Table 1, p. 12]); so the converse of Theorem A(i) and the conclusion of Theorem C were known there. Our proof puts it in case (iii), as $2 \leq x', y' \leq b - 1$, $x + y > b$ and $b < x' + y' < 2b \leq \min(ab, bc)$ exclude the rules, (T2), (T5) and (T5^R); with $q_x = a - 1$, $r = b - x'$, $q_y = c - 1$, $s = b - y'$, Theorem B gives only the support, as $z_{\max} - \lambda = x' + y' - b = 1$ and $z \geq \lambda$.

Proposition 4.13. *For the content $(2, b, 2)$, $b \geq 1$, $\mathcal{Z}(b, b) = \{b - 2j : 0 \leq j \leq b/2\}$.*

Proof. A nondecreasing vector in $\{0, 1, 2\}^b$ with sum b has as many entries 0 as 2, so $\mathcal{A}(b) = \{1 + \Lambda_m\}$ and $\mathcal{C}(b) = \{1 - \Lambda_n\}$, $0 \leq m, n \leq b/2$ ($\Lambda_0 = 0$). Each pair is realised (Lemma 2.4) and, as the Λ_m have sum 0, gives $z = b - \Lambda_m \cdot \Lambda_n = b - 2 \min(m, n)$: for $m \leq n$ the two vectors agree on the m leftmost and m rightmost positions, and Λ_m vanishes elsewhere. $\square$

For $b = 6$ the values are 6, 4, 2, 0, and 5, 3, 1 are missing. For $b \geq 2$ the cell (b, b) is (T2) ($q_x = q_y = 1$), and below the gap at $b - 1$ of Theorem C the holes go on down the fibre, one at every odd distance from the top. For $b = 2$, Proposition 4.13 is the case $i = 2$ of [19, §7, Theorem 8]; [49, Proposition 2, eq. (4.7), p. 11] gives the values $\{0, \dots, 2b\} \cup \{2t : 0 \leq t \leq 2b\}$ of z on all of $\mathcal{W}(2, b, 2)$, not on one fibre; otherwise Proposition 4.13 was not located. The cell (b, b) is on the layer of index b , whose fibre polynomials we do not take up (Table 2).

Remark 4.14. The proofs are constructive: the case of Proposition 4.6 takes $O(1)$ operations, and explicit pairs give both top values: $(A^{\text{bal}}, C^{\text{bal}})$, and its image under a move in case (iii), $(q_x + \Lambda_1, q_y - \Lambda_1)$ under (T2), or $(\Upsilon_1, c - \Upsilon_1)$ under (T5), mirrored under (T5^R). Nothing more is claimed: the multiplicity of $z_{\max} - 1$ in case (iii) is open, $z_{\max} - 2$ need not occur there (Remark 4.12), and holes can run down a whole fibre (Proposition 4.13).

5 Largest pair-count fibres and the $(2, b, 2)$ classes

A class is no larger than its fibre, so we first locate the largest ψ_2 -fibres (Theorem D): on the antidiagonal $x + y = ab$ for (a, b, a) , and exactly for $(2, b, 2)$. For $(2, b, 2)$, Theorem E then measures what splitting costs, giving the size of the largest Parikh classes for $b \geq 16$.

5.1 The antidiagonal

By Lemma 2.4, each pair $\alpha, \gamma \in \mathbb{Z}_{\geq 0}^{b+1}$ with $|\alpha| = a$ and $|\gamma| = c$ is the pair of gap counts of exactly $W(\alpha, \gamma)$ words, in which the levels of the 1s sum to x and those of the 3s to y' . Let $\mathcal{L}_n(v)$ be the set of $\alpha \in \mathbb{Z}_{\geq 0}^{b+1}$ with $\sum_g \alpha_g = n$ and $\sum_g (b-g) \alpha_g = v$. Then

$$P(x, y) = \sum_{\alpha \in \mathcal{L}_a(x)} \sum_{\gamma \in \mathcal{L}_c(y')} W(\alpha, \gamma). \quad (5.1)$$

Let $\xi = (\xi_0, \dots, \xi_b)$ be uniform on the simplex $\mathcal{T}_b$: $(\xi_0, \dots, \xi_{b-1})$ has density $b!$ on $\{\sum_{g < b} \xi_g \leq 1\} \subset \mathbb{R}_{\geq 0}^b$, and $\xi_b = 1 - \sum_{g < b} \xi_g$. With $\xi^\alpha = \prod_g \xi_g^{\alpha_g}$ and $\binom{n}{\alpha} = n! / \prod_g \alpha_g!$, the *level polynomial* $\Phi_v^{(n)}(\xi) = \sum_{\alpha \in \mathcal{L}_n(v)} \binom{n}{\alpha} \xi^\alpha$ is the coefficient of u^v in $(\xi_0 u^b + \xi_1 u^{b-1} + \dots + \xi_b)^n$.

Lemma 5.1 (Dirichlet identity). *For $m \in \mathbb{Z}_{\geq 0}^{b+1}$, $\mathbb{E}[\xi^m] = b! \prod_g m_g! / (|m| + b)!$. Hence, if $|\alpha| = a$ and $|\gamma| = c$, then $W(\alpha, \gamma) = |\mathcal{W}(a, b, c)| \binom{a}{\alpha} \binom{c}{\gamma} \mathbb{E}[\xi^{\alpha+\gamma}]$.*

Proof. Integrating out $\xi_{b-1} \in [0, r]$, $r = 1 - \xi_0 - \dots - \xi_{b-2}$, by the Beta integral $\int_0^r s^p (r-s)^q ds = r^{p+q+1} p! q! / (p+q+1)!$ (put $s = rt$ and integrate by parts) turns the integral of ξ^m over the solid simplex into $m_{b-1}! m_b! / (m_{b-1} + m_b + 1)!$ times the same integral for $b-1$ and the exponents $m_0, \dots, m_{b-2}, m_{b-1} + m_b + 1$; by induction on b it is $\prod_g m_g! / (|m| + b)!$, and the density is $b!$. For $m = \alpha + \gamma$, $|m| = a + c$, and the second right-hand side is $\prod_g (\alpha_g + \gamma_g)! / (\alpha_g! \gamma_g!) = W(\alpha, \gamma)$. $\square$

Theorem D (Largest pair-count fibres). (i) *For every content and all x, y , $P(x, y) = |\mathcal{W}(a, b, c)| \mathbb{E}[\Phi_x^{(a)}(\xi) \Phi_{y'}^{(c)}(\xi)]$.*

(ii) *Let $a = c$. The matrix $\mathbf{G}(u, v) = P(u, ab - v)$, $0 \leq u, v \leq ab$, is symmetric and positive definite. For all $0 \leq x, y \leq ab$, $P(x, y)^2 \leq P(x, ab - x) P(ab - y, y)$, with equality if and only if $x + y = ab$. Hence every largest ψ_2 -fibre of (a, b, a) lies on the antidiagonal $x + y = ab$, and $\max P = \max_x D(x)$, where $D(x) = P(x, ab - x)$.*

(iii) *Let $a = c = 2$, $h = \lfloor b/2 \rfloor$ and $N(v) = \lfloor v/2 \rfloor - \max(0, v-b) + 1$ for $0 \leq v \leq 2b$, and let $0 \leq x, y \leq 2b$, so that $y' = 2b - y$. If $x + y = 2b$, then $P(x, y) = N(x)^2 + 3N(x) + 2$ [x even]. If $x + y \neq 2b$, then, with $I_{x,y} = \mathbb{Z} \cap [\max(0, x-b, y'-b), \min(b, x, y')]$,*

$$P(x, y) = N(x) N(y') + |I_{x,y}| + [x \text{ even}, x/2 \in I_{x,y}] + [y' \text{ even}, y'/2 \in I_{x,y}].$$

The maximum of P is $(h+2)(h+3)$. For b even it is attained only at (b, b) ; for b odd it is attained exactly at $(b-1, b+1)$ and $(b+1, b-1)$, and $P(b, b) = (h+1)(h+4)$.

Proof of Theorem D(i). Sum Lemma 5.1 over $\alpha \in \mathcal{L}_a(x)$ and $\gamma \in \mathcal{L}_c(y')$, using (5.1) and $\sum_{\alpha, \gamma} \binom{a}{\alpha} \binom{c}{\gamma} \xi^{\alpha+\gamma} = \Phi_x^{(a)} \Phi_{y'}^{(c)}$. $\square$

Lemma 5.2 (Linear independence). *Let $\vartheta \in \mathbb{R}^{ab+1}$ and $F = \sum_{v=0}^{ab} \vartheta_v \Phi_v^{(a)}$. If $\mathbb{E}[F(\xi)^2] = 0$, then $\vartheta = 0$.*

Proof. As $F^2 \geq 0$ is continuous, $\mathbb{E}[F^2] = 0$ forces $F = 0$ wherever all $\xi_g > 0$, hence, F being homogeneous of degree a , on the open orthant $(0, \infty)^{b+1}$; so F is the zero polynomial. The supports $\mathcal{L}_a(v)$ of the $\Phi_v^{(a)}$ are disjoint, and nonempty for $0 \leq v \leq ab$ (put $\lfloor v/b \rfloor$ letters at level b , one at level $v - b\lfloor v/b \rfloor$ if $b \nmid v$, and the rest at level 0), and the coefficient of ξ^α , $\alpha \in \mathcal{L}_a(v)$, in F is $\vartheta_v \binom{a}{\alpha}$; so $\vartheta = 0$. $\square$

Proof of Theorem D(ii). By (i), $\mathbf{G}(u, v) = |\mathcal{W}(a, b, a)| \mathbb{E}[\Phi_u^{(a)} \Phi_v^{(a)}]$, which is symmetric, and $\vartheta^\top \mathbf{G} \vartheta = |\mathcal{W}(a, b, a)| \mathbb{E}[F^2] > 0$ for $\vartheta \neq 0$, with $F = \sum_u \vartheta_u \Phi_u^{(a)}$ (Lemma 5.2). With $u = x$ and $v = ab - y$, $\mathbf{G}(u, v) = P(x, y)$, $\mathbf{G}(u, u) = P(x, ab - x)$ and $\mathbf{G}(v, v) = P(ab - y, y)$; if $x + y = ab$, then $u = v$ and equality holds, and otherwise the strict inequality is the positivity of the principal 2×2 minor of $\mathbf{G}$ on $\{u, v\}$. A maximiser (x, y) with $x + y \neq ab$ would give $(\max P)^2 = P(x, y)^2 < P(x, ab - x)P(ab - y, y) \leq (\max P)^2$. $\square$

Remark 5.3 (What is standard, and what was not located). The counts x and y' are the Mann–Whitney statistics [18, §3, p. 51] of the 1s and the 3s against the common sample of 2s, and (i) is their standard conditional independence given the spacings ξ of that sample (cf. [35, p. 3] and [7, Proposition 2.1, eq. (2.3), p. 152]); we found no exact locator. For $a = c$, positive semidefiniteness of $\mathbf{G}$, the generic property of i.i.d. mixtures [16, §3.2, p. 9], gives the non-strict inequality in (ii), hence *some* largest ψ_2 -fibre on $x + y = ab$. Only the strict form, that *every* largest ψ_2 -fibre lies there, was not located; it is a direct consequence of Lemma 5.2, not a contribution. Placement and exact-distribution papers [10, 43, 17, 46, 11], seen only as abstracts or metadata, limit this search.

Proof of Theorem D(iii). The levels of the two 1s form a multiset $\{i, x - i\}$, $\max(0, x - b) \leq i \leq \lfloor x/2 \rfloor$, and those of the two 3s a multiset $\{j, y' - j\}$: there are $N(x)$ and $N(y')$ of them, and among those with sum v exactly one is a *double* $\{v/2, v/2\}$ if v is even, none if v is odd. By (5.1) a pair of multisets carries $W(\alpha, \gamma)$ words: 1 if the two share no level; 2, 3 or 6 if they share one level and none, one or both are doubles; and 4 if they are equal and not doubles (a double has one level, and non-doubles sharing two levels are equal). If $x = y'$, distinct multisets with sum x share no level, so $P = (N^2 - N) + 4(N - \lfloor x \text{ even} \rfloor) + 6 \lfloor x \text{ even} \rfloor = N^2 + 3N + 2 \lfloor x \text{ even} \rfloor$ with $N = N(x)$. If $x \neq y'$, no pair is equal, so a pair shares at most one level; the pairs sharing the level ℓ are $(\{\ell, x - \ell\}, \{\ell, y' - \ell\})$ with $\ell \in I_{x, y}$, and such a pair carries 3 words if $x = 2\ell$ or $y' = 2\ell$ (not both) and 2 otherwise, which gives the second formula. By (ii) every maximiser has $x = y'$, where $P = D(x)$; here $D(x) = D(2b - x)$, $N(x) = \lfloor x/2 \rfloor + 1$ for $x \leq b$, and $N \mapsto N^2 + 3N$ increases. If $b = 2h$, then $N(x) \leq h < N(b) = h + 1$ for $x < b$, so $D(b) = (h + 2)(h + 3) > (h + 1)(h + 2) \geq D(x)$ for $x \neq b$. If $b = 2h + 1$, then for $x \leq b$, $N(x) = h + 1$ exactly when $x \in \{b - 1, b\}$; so $D(b \pm 1) = (h + 2)(h + 3)$, $D(b) = (h + 1)(h + 4)$, and $D(x) \leq (h + 1)(h + 2)$ otherwise. $\square$

So D is not unimodal: $D(2i) - D(2i + 1) = 2$ whenever $2i + 1 < b$ (Figure 5). On the cells of Table 2 the totals of (iii) are [49, Theorems 2 and 6] at $a = c = 2$, directly or after reversal; elsewhere (iii), and the maximum, were not located.

Remark 5.4 ($b = 1$ and $a = 1$). Theorem D(ii) reduces (a, b, a) to maximising D . For $(a, 1, a)$, (5.1) gives $D(x) = \binom{2x}{x} \binom{2a-2x}{a-x}$; the ratio $D(x+1)/D(x) = (2x+1)(a-x)/((x+1)(2a-2x-1))$ is below, at or above 1 as x is below, at or above $(a-1)/2$, $D(x) = D(a-x)$ and $D(1)/D(0) = a/(2a-1)$. So for $a \geq 2$ the largest ψ_2 -fibres of $(a, 1, a)$ are exactly $(0, a)$ and $(a, 0)$, the two alternating classes, with $\binom{2a}{a}$ words. For $(1, b, 1)$ the 1 and the 3 share a gap on the antidiagonal, so $D(x) = 2$ and all $b + 1$ antidiagonal points tie. For $b = 2$ and $a \geq 2$ the centre $x = a$ is the unique maximum of D (Corollary 6.11); for larger even b it can lose (Proposition 5.5).

Proposition 5.5 (verified by computer). *For b even the centre $(ab/2, bc/2)$ need not be a largest ψ_2 -fibre: the largest ψ_2 -fibres of $(7, 6, 7)$ are exactly $(20, 22)$ and $(22, 20)$, with 484 422 words, against 484 020 at the centre $(21, 21)$, and those of $(5, 10, 5)$ are exactly $(24, 26)$ and $(26, 24)$, with 111 460 words, against 111 322 at the centre $(25, 25)$. In both contents the centre has the second-largest value of P . For every content with b even, $(a, c) \neq (1, 1)$ and $a + b + c \leq 19$ the centre is the unique largest ψ_2 -fibre; among such contents with $a + b + c = 20$ the only exceptions are $(7, 6, 7)$ and $(5, 10, 5)$.*

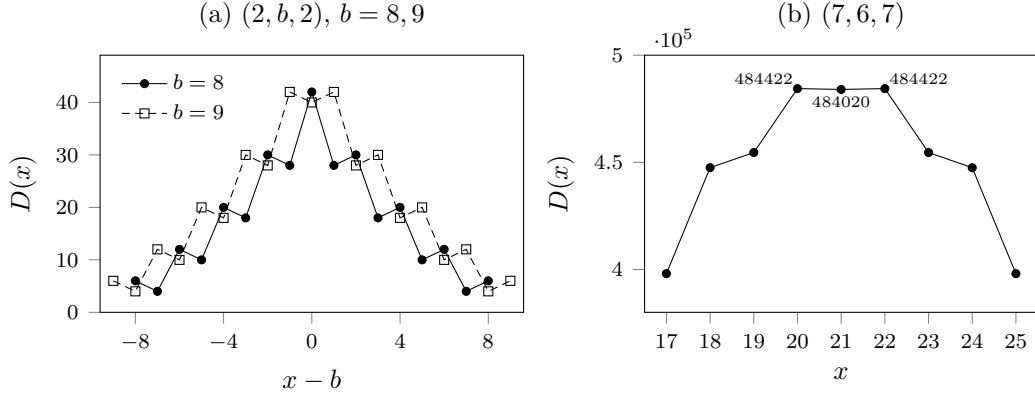


Figure 5: The fibre sizes $D(x) = P(x, ab - x)$ on the antidiagonal $x + y = ab$ of a content (a, b, a) , where every largest ψ_2 -fibre lies (Theorem D). (a) $(2, 8, 2)$ (dots) and $(2, 9, 2)$ (squares), against $x - b$. For $b = 8$ the centre $x = 8$ is the unique maximum, 42; for $b = 9$ the maximum 42 is attained at $x = 8$ and $x = 10$, and the centre has 40. (b) $(7, 6, 7)$ for $17 \leq x \leq 25$: the two largest ψ_2 -fibres of $(7, 6, 7)$, with 484 422 words at $x = 20$ and $x = 22$, exceed the centre, with 484 020.

The fibres were computed exactly by (5.1) and, for the two exceptions and (with `--full`) the whole scan, also by an independent recursion over word prefixes, in section [10] of `checks/check_antidiagonal.py`; the scan covers 436 contents with $a + b + c \leq 19$, both orders of (a, c) counted, and 516 with $a + b + c \leq 20$.

Question 5.6. *Let b be even and $(a, c) \neq (1, 1)$. Do the largest ψ_2 -fibres of (a, b, c) always lie among the centre $(ab/2, bc/2)$ and the two cells $(ab/2 \pm 1, bc/2 \mp 1)$? Is the centre the unique largest ψ_2 -fibre whenever $ab/2$ or $bc/2$ is even?*

Both answers are yes for the 3521 contents with b even, $a \leq c$, $(a, c) \neq (1, 1)$ and $a + b + c \leq 45$, hence for their mirror images (c, b, a) . Among them the centre is not the unique largest fibre exactly 30 times, from $(5, 10, 5)$ and $(7, 6, 7)$ to $(19, 6, 19)$; each time a and c are odd, $b \equiv 2 \pmod{4}$, and the two cells $(ab/2 \pm 1, bc/2 \mp 1)$ are the only largest fibres, larger than the centre by a factor between $1 + 5.4 \cdot 10^{-5}$, at $(11, 6, 19)$, and $1 + 1.9 \cdot 10^{-3}$, at $(7, 10, 7)$. The same holds in complete scans of the contents with $a \leq c$ both odd and $(a, c) \neq (1, 1)$, for $b = 6$ up to $a + b + c = 66$ (239 contents) and for $b = 10, 14, 18$ up to $a + b + c = 60$ (168, 143, 120 contents), all by exact computation (`checks/check_article2_data.py`, section [QE], `--full`).

5.2 The largest Parikh classes of $(2, b, 2)$

For $(2, b, 2)$, $\lambda = 2x + 2y - 4b$ in (2.2); put

$$d^*(b) = \left\lfloor \frac{b^2 + 9b + 36}{14} \right\rfloor, \quad V(\sigma) = \left\lfloor \frac{(\sigma + 2)^2}{4} \right\rfloor - \frac{(2\sigma - b - 2)(2\sigma - b + 1)}{2} \quad (\sigma \in \mathbb{Z}).$$

Theorem E (Largest Parikh classes of $(2, b, 2)$). (i) *For every $b \geq 16$, $d_{\max}(2, b, 2) = d^*(b)$.*

(ii) *For every $b \geq 20$, the largest Parikh classes of $(2, b, 2)$ are exactly the classes $(2X, 2Y, 0)$ with $|X - Y| \leq 1$ and $V(X + Y) = d^*(b)$, and their reversal images $(2b - 2X, 2b - 2Y, 4b - 4X - 4Y)$. The values of $X + Y$ that occur are given by Table 3, and they satisfy $|X + Y - (4b + 4)/7| < 3/4$. There are two, four or six largest classes, and six exactly when $b \equiv 0, 5, 7, 12 \pmod{14}$.*

Table 3: The largest classes of $(2, b, 2)$ for $b \geq 20$ (Theorem E(ii)). With $b = 14m + r$, $0 \leq r \leq 13$, the values of $\sigma = X + Y$ are $8m + s$ for the entries s of column r , giving the class $(\sigma, \sigma, 0)$ if σ is even and $(\sigma \mp 1, \sigma \pm 1, 0)$ if σ is odd; the middle row is the constant term of $d^*(14m + r)$.

r	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lfloor (r^2 + 9r + 36)/14 \rfloor$	2	3	4	5	6	7	9	10	12	14	16	18	20	23
s	0, 1	1	2	2	3	3, 4	4	4, 5	5	6	6	7	7, 8	8

(iii) For every $b \neq 4$ some largest Parikh class of $(2, b, 2)$ has $z = 0$ or $|w|_{321} = 0$; every largest class does if and only if $b \in \{1, 3\}$ or $b \geq 6$.

(iv) For every $b \geq 16$, $d_{\max}(2, b, 2) - (b^2 + 9b)/14$ is 2, $16/7$, $17/7$, $17/7$, $16/7$, 2, $18/7$ for $b \equiv 0, 1, 2, 3, 4, 5, 6 \pmod{7}$. In particular $d_{\max}(2, b, 2) = b^2/14 + O(b)$.

For $b \geq 20$ the proof is by hand. For $b \leq 19$, parts (i) and (iii) rest on an exhaustive exact computation, part of the proof (`checks/check_2b2.py`, sections [H] and [J]; every $b \leq 40$, and $b \leq 100$ with `--full`): the bounds of the proof do not suffice there, as the class $(8, 12, 0)$ of $(2, 16, 2)$, outside their main region, has $d^*(16) = 31$ words. For $b = 21$ the six largest classes are $(12, 12, 0)$, $(12, 14, 0)$, $(14, 12, 0)$ and their reversal images $(30, 30, 36)$, $(30, 28, 32)$, $(28, 30, 32)$; in general the largest classes with $z = 0$ have $x \approx y \approx 4(b + 1)/7$.

Proposition 5.7 (verified by computer). (i) For $b = 1, \dots, 15$ the values of $d_{\max}(2, b, 2)$ are 6, 6, 7, 8, 8, 11, 11, 13, 16, 16, 19, 22, 23, 26, 29.

(ii) For $16 \leq b \leq 19$ the largest classes are the $(x, y, 0)$ with (x, y) as follows and their reversal images: for $b = 16$ (31 words) $(8, 12)$, $(10, 10)$, $(12, 8)$; for $b = 17$ (34 words) $(10, 10)$, $(10, 12)$, $(12, 10)$; for $b = 18$ (37 words) $(10, 12)$, $(12, 10)$, $(12, 12)$; for $b = 19$ (40 words) $(10, 12)$, $(10, 14)$, $(12, 10)$, $(12, 12)$, $(14, 10)$.

(iii) For $b = 4$ the only largest class is $(4, 4, 4)$, with 8 words. For $b = 2$ the class $(2, 2, 2)$, and for $b = 5$ the classes $(4, 6, 4)$, $(5, 5, 5)$, $(6, 4, 4)$, which have $z \geq 1$ and $|w|_{321} \geq 1$, tie with the largest classes with $z = 0$, at 6 and 8 words.

This is the enumeration of `checks/check_2b2.py` (sections [B], [H], [J]), confirmed by an independent one in `checks/check_article2_data.py` (section [B], $b \leq 20$, and $b \leq 40$ with `--full`).

Corollary 5.8. $d_{\max}(2, b, 2) = d^*(b)$ if and only if $b \in \{10, 13\}$ or $b \geq 16$.

Proof. For $b \geq 16$ this is Theorem E(i); for $b \leq 15$ compare Proposition 5.7(i) with $d^*(b)$: equality holds at $d^*(10) = 16$ and $d^*(13) = 23$, and $d^*(b) < d_{\max}(2, b, 2)$ for the other $b \leq 15$. $\square$

In the direction of the question “How much ambiguity does a given Parikh matrix leave about a word?” [22, §8, p. 19], Theorem E gives the largest degree of ambiguity [34, Definition 6, p. 109] at content $(2, b, 2)$. It was not located; the nearest data are the 210 words of $(2, 3, 2)$ in 70 classes [2, Example 19, p. 106] and the values of z on $\mathcal{W}(2, b, 2)$, without multiplicities [49, Proposition 2, eq. (4.7), p. 11].

5.3 Proof of Theorem E

In a word of content $(2, b, 2)$ let $u_1 \leq u_2$ be the levels of the two 1s and $v_1 \leq v_2$ those of the two 3s, and write $z' = |w|_{321}$; in the lists p_i, q_j of Section 2 the levels are the $b - p_i$ and the q_j .

Lemma 5.9 ([49, Lemma 1, eqs. (2.5)–(2.6), (2.8), p. 3]). *We have $x = u_1 + u_2$, $y' = v_1 + v_2$, $z = \sum_{i,j} (b - p_i - q_j)_+ = \sum_{i,j} (u_i - v_j)_+$, $z' = \sum_{i,j} (p_i + q_j - b)_+ = \sum_{i,j} (v_j - u_i)_+$, and $z - z' = 2x - 2y'$.*

Proof. A letter of level u precedes u letters 2, which gives x and y' . A 1 of level u and a 3 of level v form $u - v \geq 0$ occurrences of 123, through the 2s between them, and none of 321 if the 1 is to the left, and $v - u \geq 0$ occurrences of 321 and none of 123 otherwise; sum over the four pairs. $\square$

By (2.2), reversal maps (x, y, z, z') to $(2b - x, 2b - y, z', z)$, and so $x - y'$ to $y' - x$; thus $\rho(x, y, z) = \rho(2b - x, 2b - y, z')$. Call a class *thin* if $z \geq 1$ and $z' \geq 1$; every other class has $z = 0$ or is the reversal image of one with $z = 0$, so $d_{\max}(2, b, 2)$ is the larger of the largest thin class and $\max_{x,y} \rho(x, y, 0)$. For fixed (x, y) the configurations of levels are the points (u_2, v_1) with $\lceil x/2 \rceil \leq u_2 \leq \min(x, b)$ and $\max(0, y' - b) \leq v_1 \leq \lfloor y'/2 \rfloor$, where $u_1 = x - u_2$ and $v_2 = y' - v_1$; each coordinate takes at most $\lfloor b/2 \rfloor + 1$ values, as $\min(x, b) - \lceil x/2 \rceil \leq \lfloor b/2 \rfloor$, and likewise for y' . By Lemma 2.4, $\rho(x, y, z)$ is the sum of the weights $W(\alpha, \gamma)$ of the points with $\sum_{i,j} (u_i - v_j)_+ = z$, and a weight exceeds 1 exactly when a 1 and a 3 share a level.

Lemma 5.10 (Thin classes). *If $z \geq 1$ and $z' \geq 1$, then $\rho(x, y, z) \leq b + 4$. For b even, $\rho(b, b, b) = b + 4$.*

Proof. By reversal we may assume $x \leq y'$; then $z' = z + 2(y' - x) \geq z =: \delta \geq 1$. As $u_1 \leq x/2 \leq y'/2 \leq v_2$, and $u_1 > v_1$ and $u_2 > v_2$ together would give $x > y'$, Lemma 5.9 gives $z = 0$ if $u_2 \leq v_1$, and otherwise $z = (u_2 - v_1) + \text{dist}(\kappa, [x, y'])$ with $\kappa = u_2 + v_1$, as $u_1 \leq v_1$ if and only if $x \leq \kappa$, and $u_2 \leq v_2$ if and only if $\kappa \leq y'$. Fix u_2 and let $v_1 < u_2$ grow: each step lowers $u_2 - v_1$ by 1 and changes the distance by $-1, 0$ or 1 as $\kappa < x$, $x \leq \kappa < y'$ or $\kappa \geq y'$. So z strictly decreases while $\kappa < y'$, where $z \geq u_2 - v_1 > 2u_2 - y'$, and $z = 2u_2 - y'$ once $\kappa \geq y'$. Hence each u_2 carries at most one point with $z = \delta$, except $u_2 = (y' + \delta)/2$, which carries at most $\lfloor b/2 \rfloor + 1$; so at most $2\lfloor b/2 \rfloor + 1 \leq b + 1$ points have $z = \delta$. On them $u_2 > v_1$, and $u_1 = v_2$ would give $u_2 - v_1 = x - y' \leq 0$; so a shared level means $u_1 = v_1$, that is $\kappa = x$, or $u_2 = v_2$, that is $\kappa = y'$, and then $z = u_2 - v_1 = \delta$ fixes the point: there is at most one of each, and they coincide if and only if $x = y'$. If $x < y'$, each has weight 2: at $u_1 = v_1$ we have $u_2 \neq u_1$ (as $u_2 > v_1$), $v_2 \neq v_1$ (else $y' = 2u_1 \leq x$) and $u_2 \neq v_2$ (else $x = y'$); at $u_2 = v_2$ we have $u_1 \neq u_2$ (else $x = 2u_2 > \kappa = y'$) and $v_1 \neq v_2$ (as $v_1 < u_2$). If $x = y'$, the only such point has $u_1 = v_1 < u_2 = v_2$ and weight 4. So the weights exceed the number of points by at most 3, and $\rho(x, y, z) \leq b + 4$. For b even, $x = y = b$ and $\delta = b$: if $\kappa \geq b$, then $z = 2u_2 - b = b$ forces $u_2 = b$, and if $\kappa < b$, then $z = b - 2v_1 = b$ forces $v_1 = 0$; so the points are the (b, v_1) with $0 \leq v_1 \leq b/2$ and the $(u_2, 0)$ with $b/2 \leq u_2 < b$. The point $(b, 0)$ has weight 4 and the other b points weight 1, so $\rho(b, b, b) = b + 4$. $\square$

Lemma 5.11 (Classes with $z = 0$). *Let $X = \lfloor x/2 \rfloor$, $Y = \lfloor y/2 \rfloor$, $t = x + y - b$, and let $B_{\geq t}$ and $B_{=t}$ be the sets of $(i, j) \in \mathbb{Z}^2$ with $0 \leq i \leq X$, $0 \leq j \leq Y$ and $i + j \geq t$, respectively $i + j = t$. Then*

$$\begin{aligned} \rho(x, y, 0) = & |B_{\geq t}| + |B_{=t}| + [x \text{ even}, X \leq t \leq X + Y] \\ & + [y \text{ even}, Y \leq t \leq X + Y] + 2[x, y \text{ even}, t = X + Y]. \end{aligned}$$

The classes with $z' = 0$ are the reversal images: $\rho(x, y, \lambda) = \rho(2b - x, 2b - y, 0)$. For $t \leq 0$, and for $t = 1$ with $x, y \geq 2$, these values are [49, Theorems 2 and 6] at $a = c = 2$, after reversal.

Proof. By Lemma 5.9, $z = 0$ if and only if $u_2 \leq v_1$. Put $i = u_1$ and $j = b - v_2$; then $u_2 = x - i$, $v_1 = b - y + j$, and $u_2 \leq v_1$ if and only if $i + j \geq t$. The conditions $0 \leq i \leq X$ and $0 \leq j \leq Y$

say that $0 \leq u_1 \leq u_2$ and $v_1 \leq v_2 \leq b$, and with $u_2 \leq v_1$ they give $u_2 \leq b$ and $v_1 \geq 0$; so the configurations with $z = 0$ are the points of $B_{\geq t}$. On them $u_1 \leq u_2 \leq v_1 \leq v_2$, so a 1 and a 3 can share only the level $u_2 = v_1$, which happens if and only if $i + j = t$. That level holds two 1s if $u_1 = u_2$, that is if x is even and $i = X$, and two 3s if $v_1 = v_2$, that is if y is even and $j = Y$; its factor in $W(\alpha, \gamma)$ is $2 + [u_1 = u_2] + [v_1 = v_2] + 2[u_1 = u_2, v_1 = v_2]$, that is, 2, 3 or 6. All other factors are 1, and $(X, t - X)$, $(t - Y, Y)$ and (X, Y) lie in $B_{=t}$ exactly under the conditions in the brackets. The second statement is reversal, as $z - z' = \lambda$. $\square$

Remark 5.12 (The bottom of a fibre). The fibre (x, y) of $(2, b, 2)$ lies on the layer of index $3b - x - y$ of [49], and its bottom, the class $z = \lambda$ with $|w|_{321} = 0$ (nonempty only if $x + y \geq 2b$), is counted by the second statement of Lemma 5.11; on the exterior layers of indices $2, \dots, b$ this lies in the further direction named in [49, §7, item 2, p. 14], exact fibre polynomials on those layers. For instance, at (b, b) , on the layer of index b , $\lambda = 0$ and the bottom coefficient is $\rho(b, b, 0) = 6$ if b is even; if b is odd there is no class $z = \lambda$. We use these counts only to locate the largest classes of $(2, b, 2)$, and give no fibre polynomial on these layers in general: there we only bound the thin classes, with $\rho(b, b, b)$ for even b (Lemma 5.10), and give the totals (Theorem D(iii)), the values at (b, b) (Proposition 4.13) and, at the exceptional cells, the multiplicity of $z_{\max} - 2$ (Proposition 4.10), which at $x = 2$ or $y = 2$ is also the bottom coefficient. For $b \geq 16$ no largest class lies on the cells credited to [49]: the largest classes with $z = 0$ have $t \geq 3$ (Proposition 5.7 for $b \leq 19$, the proof below for $b \geq 20$), and their reversal images lie on the layers of index $t \geq 3$.

By the symmetry $(x, X) \leftrightarrow (y, Y)$ of Lemma 5.11 we may assume $X \leq Y$. Put $\sigma = X + Y$ and $\chi = [x \text{ odd}] + [y \text{ odd}]$, so $t = 2\sigma - b + \chi$, and $e = X + Y - t$.

Lemma 5.13 (Piecewise form). *Let $X \leq Y$. Then $|B_{\geq t}| + |B_{=t}|$ is $(X + 1)(Y + 1)$ in region (I), $t \leq -1$; $(X + 1)(Y + 1) - (t - 2)(t + 1)/2$ in (II), $0 \leq t \leq X - 1$; $(X + 1)(Y - t + 2 + X/2)$ in (III), $X \leq t \leq Y$; $(e + 1)(e + 4)/2$ in (IV), $Y \leq t \leq X + Y$; and 0 in (V), $t > X + Y$. The brackets of Lemma 5.11 vanish in (I), (II) and (V).*

Proof. In (II) the points with $i + j \leq t - 1$ form a full triangle of $t(t + 1)/2$ points, and $|B_{=t}| = t + 1$, as $t - 1 < X \leq Y$. In (III), for each $i \leq X$ the index j runs over $[t - i, Y]$, as $0 \leq t - i \leq Y$, giving $(X + 1)(Y - t + 1) + X(X + 1)/2$ points, and $|B_{=t}| = X + 1$. In (IV), $i + j \geq t$ if and only if $(X - i) + (Y - j) \leq e$, with $e \leq X \leq Y$: a full triangle of $(e + 1)(e + 2)/2$ points, and $|B_{=t}| = e + 1$. (I) and (V) are clear, and the brackets need $t \geq X$, $t \geq Y$ or $t = X + Y$, with $t \leq X + Y$. $\square$

Lemma 5.14 (Bounds for $z = 0$). *Let $X \leq Y$.*

- (a) *If $t \leq -1$, or $0 \leq t \leq X - 1$ and $2\sigma < b$, then $\rho(x, y, 0) \leq (b + 3)^2/16 + 1$.*
- (b) *If $0 \leq t \leq X - 1$ and $2\sigma \geq b$, then $\rho(x, y, 0) \leq V(\sigma)$, with equality if and only if $|X - Y| \leq 1$ and either $\chi = 0$, or $2\sigma = b$ and $\chi = 1$.*
- (c) *If $t \geq X$, then $\rho(x, y, 0) \leq \max((b + 6)^2/16 + 1, (b + 3)(b + 12)/18 + 2, 6)$.*

Proof. The function $t \mapsto (t - 2)(t + 1)/2$ increases by t from t to $t + 1$: it is -1 at $t = 0, 1$, and strictly increasing from $t = 1$ on. For $X + Y = \sigma$, $(X + 1)(Y + 1) = ((\sigma + 2)^2 - (X - Y)^2)/4 \leq \lfloor (\sigma + 2)^2/4 \rfloor$, with equality if and only if $|X - Y| \leq 1$. (a) In both cases $2\sigma \leq b - 1$ (in (I) as $2\sigma = t + b - \chi$), so $(X + 1)(Y + 1) \leq (\sigma + 2)^2/4 \leq (b + 3)^2/16$; in (I) $\rho(x, y, 0) = (X + 1)(Y + 1)$, and in (II) $t \leq (b - 1) + 2 - b = 1$, so $\rho(x, y, 0) = (X + 1)(Y + 1) + 1$. (b) As $t = (2\sigma - b) + \chi$ with $2\sigma - b \geq 0$, $(t - 2)(t + 1)/2$ is at least its value at $2\sigma - b$, with equality if and only if $\chi = 0$ or $(2\sigma - b, \chi) = (0, 1)$; with the product bound this gives (b). (c) In (III) with $t < Y$ the brackets reduce to $[x \text{ even}]$, and $t \geq X$ gives $Y \geq (b - \chi - X)/2$, so $|B_{\geq t}| + |B_{=t}| = (X + 1)(b - \chi + 2 - 3X/2 - Y) \leq (X + 1)((b - \chi)/2 + 2 - X)$; as $\chi \geq 1$

if x is odd, $\rho(x, y, 0) \leq (X + 1)(b/2 + 2 - X) + 1 \leq (b + 6)^2/16 + 1$, the two factors adding up to $b/2 + 3$. At $t = Y$ the count is $(X + 1)(X + 4)/2$ and the brackets add up to at most 4 if $X = 0$ and to at most 2 otherwise; and $Y = b - \chi - 2X \geq X$ gives $3X \leq b$, so $\rho(x, y, 0) \leq 6$ or $\rho(x, y, 0) \leq (b + 3)(b + 12)/18 + 2$. In (IV) with $t > Y$ we have $e = b - \sigma - \chi$ and $e \leq X - 1 \leq \sigma/2 - 1$, so $3e \leq b - 2$; the brackets add up to at most 2 unless $e = 0$, where the count is 2 and they add up to at most 4; so $\rho(x, y, 0) \leq 6$ or $\rho(x, y, 0) \leq (b + 1)(b + 10)/18 + 2 < (b + 3)(b + 12)/18 + 2$. In (V) there is no word. $\square$

Lemma 5.15 (The parabola). *For every $b \geq 1$ and $\sigma \in \mathbb{Z}$,*

$$V(\sigma) \leq \frac{b^2 + 9b + 36}{14} - \frac{7}{4} \left(\sigma - \frac{4b + 4}{7} \right)^2.$$

Hence $V(\sigma) \leq d^*(b)$, and $V(\sigma) = d^*(b)$ implies $|\sigma - (4b + 4)/7| < 3/4$.

Proof. The quadratic $(\sigma + 2)^2/4 - (2\sigma - b - 2)(2\sigma - b + 1)/2$ has leading coefficient $-7/4$ and derivative $(4b + 4 - 7\sigma)/2$, vanishing at $\sigma^* = (4b + 4)/7$, where $\sigma^* + 2 = (4b + 18)/7$, $2\sigma^* - b = (b + 8)/7$, and its value is $(4b + 18)^2/196 - (b^2 + 9b - 90)/98 = (b^2 + 9b + 36)/14$. This and $\lfloor (\sigma + 2)^2/4 \rfloor \leq (\sigma + 2)^2/4$ give the inequality. $V(\sigma)$ is an integer, as $2\sigma - b - 2$ and $2\sigma - b + 1$ have opposite parity; so $V(\sigma) \leq d^*(b)$. As $d^*(b) \geq (b^2 + 9b + 23)/14$, $V(\sigma) = d^*(b)$ gives $(\sigma - \sigma^*)^2 \leq 26/49 < 9/16$. $\square$

Proof of Theorem E. Let $b \geq 20$. A thin class has at most $b + 4$ words (Lemma 5.10), and a class with $z = 0$ in case (a) or (c) of Lemma 5.14 at most $(b + 3)^2/16 + 1$, $(b + 6)^2/16 + 1$, $(b + 3)(b + 12)/18 + 2$ or 6. Each of these bounds is below $(b^2 + 9b + 23)/14 \leq d^*(b)$: for 6 this is clear, and for the others since $b^2 - 5b - 33$, $8(b^2 + 9b + 23) - 7(b + 3)^2 - 112 = b^2 + 30b + 9$, $8(b^2 + 9b + 23) - 7(b + 6)^2 - 112 = b^2 - 12b - 180$ and $9(b^2 + 9b + 23) - 7(b + 3)(b + 12) - 252 = 2b^2 - 24b - 297$ are positive, except $(b + 6)^2/16 + 1$ at $b = 20$, which is $26^2/16 + 1 < 44 = d^*(20)$. So these classes and their reversal images have fewer than $d^*(b)$ words, and in case (b) $\rho(x, y, 0) \leq V(\sigma) \leq d^*(b)$ by Lemma 5.15. Equality there requires $|X - Y| \leq 1$, $V(\sigma) = d^*(b)$, and $\chi = 0$ or $2\sigma = b$; the latter is impossible, as $|\sigma - (4b + 4)/7| < 3/4$ while $|b/2 - (4b + 4)/7| = (b + 8)/14 \geq 2$. So $x = 2X$ and $y = 2Y$ are even, and $t = 2\sigma - b > (b + 8)/7 - 3/2 > 2$, as claimed in Remark 5.12.

Conversely, let $b = 14m + r$ and $\sigma = 8m + s$ with $0 \leq r \leq 13$, and $x = 2X$, $y = 2Y$ with $X = \lfloor \sigma/2 \rfloor$, $Y = \lceil \sigma/2 \rceil$ (or the reverse). Expanding, $V(\sigma)$ and $d^*(b)$ are $14m^2 + (2r + 9)m$ plus, respectively, $(\lfloor s/2 \rfloor + 1)(\lceil s/2 \rceil + 1) - (2s - r - 2)(2s - r + 1)/2$ and $\lfloor (r^2 + 9r + 36)/14 \rfloor$; so $V(\sigma) - d^*(b)$ depends only on (r, s) . By Lemma 5.15 it vanishes only if $|s - (4r + 4)/7| < 3/4$, which leaves at most two s for each r ; at these pairs it is 0 exactly at the entries of Table 3, and -1 at the other four, $(r, s) = (2, 1), (3, 3), (9, 5)$ and $(10, 7)$ (checks/check_2b2.py, section [G], recomputes the zeros). The point lies in case (b), $0 \leq t \leq X - 1$ with $t = 2m + 2s - r$, if and only if $m \geq (r - 2s)/2$ and $m \geq (2s - r - \lfloor s/2 \rfloor + 1)/2$; over the entries of the table the first bound is at most 0, the second at most $3/2$, and at most 1 if $r \geq 6$; and $b \geq 20$ means $m \geq 2$ if $r \leq 5$ and $m \geq 1$ if $r \geq 6$. There $\rho(x, y, 0) = V(\sigma) = d^*(b)$. This proves (i) and the first two sentences of (ii) for $b \geq 20$. An even σ gives one class with $z = 0$ and an odd σ two, and their reversal images, with $z = 4b - 4\sigma > 0$, are different classes; so there are two, four or six largest classes, six exactly when $r \in \{0, 5, 7, 12\}$.

For $16 \leq b \leq 19$, (i) is the computation of checks/check_2b2.py, section [H], which gives $d_{\max}(2, b, 2) = d^*(b)$, namely 31, 34, 37 and 40. Part (iii) follows from (ii) for $b \geq 20$, and for $b \leq 19$ it is the computation of section [J]. For (iv), (i) gives $d_{\max}(2, b, 2) - (b^2 + 9b)/14 = \lfloor (b(b + 9)/2 + 18)/7 \rfloor - b(b + 9)/14$, which depends only on the residue of the integer $b(b + 9)/2$ modulo 7; as $(b + 7)(b + 16)/2 - b(b + 9)/2 = 7b + 56$, that residue depends only on b modulo 7, and $b = 0, \dots, 6$ give the seven values. $\square$

6 The golden-ratio threshold and the largest ψ_2 -fibre of $(a, 2, c)$

With two letters 2, a word is a lattice path with two marked points, and fibres and classes are sums of path counts. The largest ψ_2 -fibre is the centre of the box (Theorem G), which in general splits into several classes (Figure 1), while some largest Parikh class lies on the boundary; for $a \leq c$ it is no larger than the largest trace class exactly when $c \leq c^*(a) \sim \varphi a$ (Theorem F).

Throughout, the content is $(a, 2, c)$ with $a, c \geq 1$; x, y, z, ρ, P and $\Delta = xy - 2z \geq 0$ are as in Section 2 with $b = 2$, and the box is $0 \leq x \leq 2a, 0 \leq y \leq 2c$. We put $\tau = \binom{a+c}{a}$, $\beta(a, c) = d_{\max}(a, 2, c)$ and

$$P_0(m) = P(0, 2c - m) \quad (0 \leq m \leq 2c), \quad f(j) = \sum_{i=j}^{\min(2j, c)} \binom{a+c-i}{a} \quad (0 \leq j \leq c); \quad (6.1)$$

P'_0 is P_0 for the content $(c, 2, a)$, and j^* is the least j at which f is maximal. As the gap occupancies fix where the letters 2 go (Lemma 2.4), a word of a trace class is determined by its $\{1, 3\}$ -subword, so a trace class has at most τ words; the three τ -classes $\{22u\}$, $\{u22\}$, $\{2u2\}$, where u runs over the words with a letters 1 and c letters 3, have exactly τ words (cf. [34, pp. 110–111], with no letter 2 and $a = c$); their triples are $(0, 2c, 0)$, $(2a, 0, 0)$, $(a, c, 0)$.

6.1 Lattice paths with two marked points

Lemma 6.1. *Read the $\{1, 3\}$ -subword ω of a word w of content $(a, 2, c)$ as a lattice path from $(0, 0)$ to (a, c) , with an east step for each 1 and a north step for each 3. For $l = 1, 2$ let A_l and C_l be the numbers of letters 1 before and of letters 3 after the l -th letter 2, and $v_l = (A_l, c - C_l)$. Then $w \mapsto (\omega, v_1, v_2)$ is a bijection onto the triples of a path ω and two of its lattice points v_1, v_2 , with v_1 not after v_2 along ω (possibly $v_1 = v_2$); and with $(\delta, \varepsilon) = v_2 - v_1$, we have $x = A_1 + A_2$, $y = C_1 + C_2$, $z = A_1C_1 + A_2C_2$ and $\Delta = \delta\varepsilon$.*

Proof. The l -th letter 2 sits at the point v_l of ω , whose coordinates count the letters 1 and 3 before it, and inserting letters 2 at two such points recovers w . A letter 2 forms an occurrence of 12 with each 1 before it, of 23 with each 3 after it, and of 123 with each pair of these; and $(A_1 + A_2)(C_1 + C_2) - 2A_1C_1 - 2A_2C_2 = (A_2 - A_1)(C_1 - C_2)$. $\square$

Lemma 6.1 restates the profile model (Lemma 2.4) for $b = 2$ (Figure 6); $\Delta = \delta\varepsilon$, the case $b = 2$ of (2.3) [22, §7, p. 15], counts the pairs of a 1 and a 3 between the two 2s. The words with prescribed points $v_1 \leq v_2$ are the paths through both, so $\rho(x, y, z)$ is the sum of $\binom{A_1+c-C_1}{A_1} \binom{\delta+\varepsilon}{\delta} \binom{a-A_2+C_2}{C_2}$ over the comparable pairs $v_1 \leq v_2$ with $v_1 + v_2 = (x, 2c - y)$ and $\delta\varepsilon = \Delta$; this elementary consequence was not located, and is not claimed.

The *centre* of a class is $(\mu, \nu) = \frac{1}{2}(v_1 + v_2) = (x/2, c - y/2) \in (\frac{1}{2}\mathbb{Z})^2$: the pair (x, y) fixes it, and then Δ fixes the class (the centre (a, c) of the box gives $(\mu, \nu) = (a/2, c/2)$). A fibre or class is *interior* if $0 < x < 2a$ and $0 < y < 2c$, and on the *boundary* otherwise. For an interior centre, $K_{a,c}(\mu, \nu) = \rho(2\mu, 2c - 2\nu, 2\mu(c - \nu))$ is the size of the class with centre (μ, ν) on the *Cauchy surface* $\Delta = 0$, where $z \leq xy/2$ (Lemma 2.5) is attained.

Lemma 6.2. (i) $\Delta = 0$ if and only if $\delta = 0$ or $\varepsilon = 0$, that is, v_1, v_2 lie on a common vertical line (then $\mu \in \mathbb{Z}$) or horizontal line (then $\nu \in \mathbb{Z}$). If μ and ν are half-integers, then $\Delta \geq 1$. (ii) Every boundary ψ_2 -fibre is one Parikh class: $z = 0$ if $x = 0$ or $y = 0$, $z = ay$ if $x = 2a$, and $z = cx$ if $y = 2c$. (iii) For an interior centre, $K_{a,c}(\mu, \nu) = K^v(\mu, \nu) + K^h(\mu, \nu)$, where

$$K^v(\mu, \nu) = [\mu \in \mathbb{Z}] \sum_{i=\lceil \nu \rceil}^{\min(2\nu, c)} \binom{\mu+2\nu-i}{\mu} \binom{a-\mu+c-i}{a-\mu}, \quad K^h(\mu, \nu) = [\nu \in \mathbb{Z}] \sum_{j=\lceil \mu \rceil+1}^{\min(2\mu, a)} \binom{2\mu-j+\nu}{\nu} \binom{a-j+c-\nu}{c-\nu}.$$

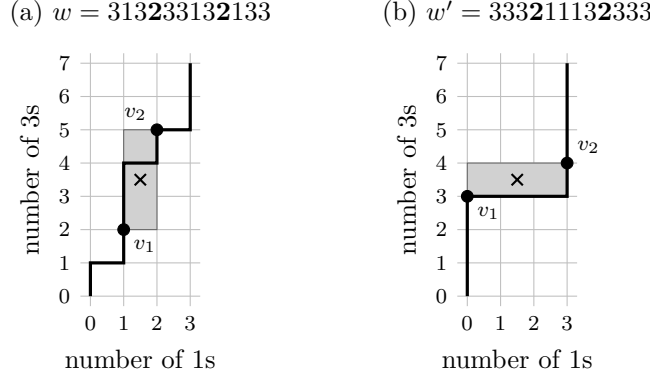


Figure 6: The lattice-path model for content $(a, 2, c) = (3, 2, 7)$. Deleting the two letters 2 of a word leaves its $\{1, 3\}$ -subword ω , read as a lattice path from $(0, 0)$ to (a, c) in which a 1 is an east step and a 3 a north step; the l -th letter 2 sits at a lattice point v_l of ω , and $v_1 \leq v_2$. The abscissae of v_1 and v_2 add up to x , their ordinates add up to $2c - y$, and with $(\delta, \varepsilon) = v_2 - v_1$ the number $\Delta = xy - 2z = \delta\varepsilon$ counts the pairs of a 1 and a 3 between the two 2s (shaded). (a) $v_1 = (1, 2)$, $v_2 = (2, 5)$, $\Delta = 1 \cdot 3$. (b) $v_1 = (0, 3)$, $v_2 = (3, 4)$, $\Delta = 3 \cdot 1$. Both words have $(x, y, z) = (3, 7, 9)$ and the centre $(\mu, \nu) = (3/2, 7/2)$ (cross). Their class consists of the words whose pair (v_1, v_2) is one of these two, and has 40 words (Figure 1(b)).

Proof. (i) is the case $b = 2$ of the equality case in Lemma 2.5 [22, Lemma 2], as $\Delta = \delta\varepsilon$; and δ, ε are odd if μ, ν are half-integers. (ii) is rule (E) of Theorem A(i): $x \in \{0, 2a\}$ forces $A_1 = A_2 \in \{0, a\}$, so $z = A_1y$, and $y \in \{0, 2c\}$ forces $C_1 = C_2 \in \{0, c\}$, so $z = C_1x$. (iii) By (i) the class consists of the vertical pairs $v_1 = (\mu, 2\nu - i)$, $v_2 = (\mu, i)$ and the horizontal pairs $v_1 = (2\mu - j, \nu)$, $v_2 = (j, \nu)$ with $j > \mu$ (a pair $v_1 = v_2$ counts as vertical); a path through a vertical pair runs from $(0, 0)$ to the lower point, straight up, and on to (a, c) ; similarly for horizontal pairs. $\square$

By Lemma 2.3, reversal, the complement and the mirror map keep Δ and permute the boundary fibres of $(a, 2, c)$ and $(c, 2, a)$. Reversal maps (x, y) to $(2a - x, 2c - y)$, so (μ, ν) to $(a - \mu, c - \nu)$, and the complement maps $(a, c; x, y)$ to $(c, a; 2c - y, 2a - x)$, so $(a, c; \mu, \nu)$ to $(c, a; \nu, \mu)$:

$$K_{a,c}(\mu, \nu) = K_{a,c}(a - \mu, c - \nu) = K_{c,a}(\nu, \mu). \quad (6.2)$$

6.2 The golden-ratio threshold

Theorem F (The golden-ratio threshold). *Let $1 \leq a \leq c$ (for $a > c$, use the mirror map).*

- (i) $P_0(m) = \sum_{i=\lceil m/2 \rceil}^{\min(m,c)} \binom{a+c-i}{a}$ for $0 \leq m \leq 2c$; $P(2a, y) = P_0(y)$, $P(x, 0) = P'_0(2a - x)$ and $P(x, 2c) = P'_0(x)$. Every boundary ψ_2 -fibre is a single Parikh class.
- (ii) $f(0) = \tau$, and $f(0), \dots, f(c)$ strictly increases up to j^* and strictly decreases after it, except that $f(j^* + 1) = f(j^*)$ may occur; j^* is the least $j \geq 0$ with $\binom{a+c-2j-1}{a} + \binom{a+c-2j-2}{a} \leq \binom{a+c-j}{a}$.
- (iii) $\beta(a, c) = \max_j f(j) = f(j^*)$, the size of the largest boundary fibre. A class with $\Delta > 0$ has at most τ words, with equality only for $(a, c, 0)$. An interior class with $\Delta = 0$ has fewer than $\beta(a, c)$ words, except the class $(x, y, z) = (2, 2, 2)$ of the content $(2, 2, 2)$, which has $\beta(2, 2)$.
- (iv) $\beta(a, c) = \tau$ if and only if $c(c - 1) \leq a(a + c - 1)$, if and only if $c \leq c^*(a) = ((a + 1) + \sqrt{5a^2 - 2a + 1})/2$; and $c^*(a) - \varphi a \in ((1 - 1/\sqrt{5})/2, (3 - \sqrt{5})/2] = (0.276 \dots, 0.381 \dots]$. If $c(c - 1) > a(a + c - 1)$, then $\beta(a, c) = f(j^*) > \tau$.

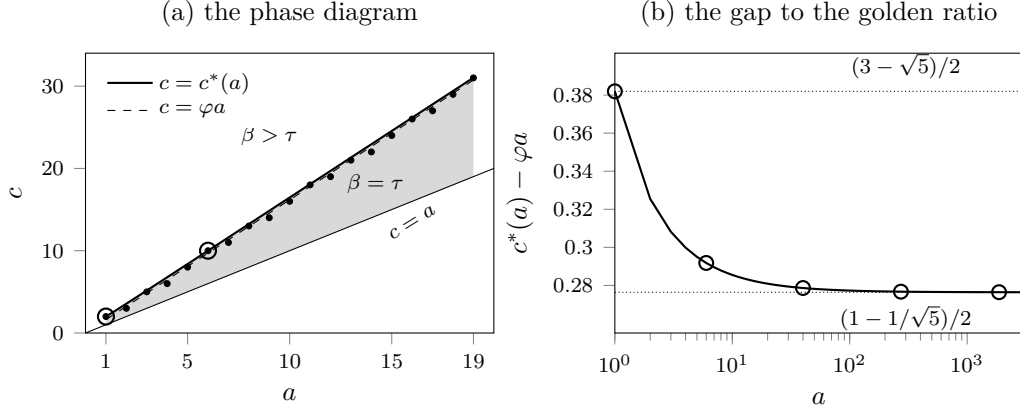


Figure 7: The golden-ratio threshold for contents $(a, 2, c)$ with $1 \leq a \leq c$ (Theorem F). (a) In the shaded region $a \leq c \leq c^*(a) = ((a+1) + \sqrt{5a^2 - 2a + 1})/2$ the largest Parikh class has $\beta(a, c) = \tau = \binom{a+c}{a}$ words; above the curve $\beta(a, c) > \tau$. Dots: the largest c with $\beta(a, c) = \tau$ for $a = 1, \dots, 19$, namely $2, 3, 5, 6, 8, \dots, 29, 31$. Circles: the ties $(1, 2)$ and $(6, 10)$, where $c(c-1) = a(a+c-1)$ and five classes have τ words. (b) The gap $c^*(a) - \varphi a$, on a logarithmic a -axis, lies in $((1 - 1/\sqrt{5})/2, (3 - \sqrt{5})/2]$ (dotted). Circles: the ties $(F_n F_{n+1}, F_n F_{n+2})$ for $n = 1, 3, 5, 7, 9$, at $a = 1, 6, 40, 273, 1870$.

- (v) *The largest Parikh classes are exactly: if $c(c-1) < a(a+c-1)$, the τ -classes $(0, 2c, 0)$, $(2a, 0, 0)$, $(a, c, 0)$, and also $(2, 2, 2)$ when $a = c = 2$; if $c(c-1) = a(a+c-1)$, the classes $(0, 2c, 0)$, $(0, 2c-2, 0)$, $(2a, 0, 0)$, $(2a, 2, 2a)$, $(a, c, 0)$; if $c(c-1) > a(a+c-1)$, the classes $(0, 2c-2j^*, 0)$ and $(2a, 2j^*, 2aj^*)$, together with $(0, 2c-2j^*-2, 0)$ and $(2a, 2j^*+2, 2a(j^*+1))$ when $f(j^*+1) = f(j^*)$.*
- (vi) *$c(c-1) = a(a+c-1)$ if and only if $(a, c) = (F_n F_{n+1}, F_n F_{n+2})$ for an odd $n \geq 1$, that is, $(a, c) = (1, 2), (6, 10), (40, 65), (273, 442), (1870, 3026), \dots$; for these contents exactly five classes have $\beta(a, c) = \tau$ words.*

For example, $(3, 2, 7)$ has $\beta(3, 7) = f(1) = 140 > \tau = 120$, attained by $(0, 12, 0)$ and $(6, 2, 6)$ (Figure 1); $(6, 2, 10)$ is a tie, with five largest classes of 8008 words; $(2, 2, 2)$ has the four largest classes $(0, 4, 0)$, $(2, 2, 0)$, $(2, 2, 2)$, $(4, 0, 0)$, of 6 words. Among the 1225 contents with $a + c \leq 50$, in both orders, the number of largest classes is 2 for 878 contents, 3 for 310, 4 for 33 and 5 for 4, as (v) predicts (exact computation of every class, for $a > c$ through the mirror map: `checks/check_r7_golden.py`, section [3]). Figure 7 shows the integer thresholds $\lfloor c^*(a) \rfloor$; the proof (Section 6.5) uses a finite certificate (Appendix A).

Parts (i) and (ii) are elementary, and the tie set in (vi) is classical: by Corollary 6.5(i), $c(c-1) = a(a+c-1)$ if and only if $\binom{a+c-2}{a} = \binom{a+c-1}{a-1}$, which for $a \geq 2$ is Singmaster's equation $\binom{n+1}{k+1} = \binom{n}{k+2}$ at $(n, k) = (a+c-2, a-2)$, solved through the same Pell equation in [39, §2, pp. 295–296]; the OEIS records its solutions [27] and the coordinates of the ties [26, 25]. So Lemma 6.10 is known; we prove it for completeness. Parts (iii)–(v) and the five largest classes in (vi) were not located: Salomaa's study of $d(\Psi)$ [34, §5] does not treat $(a, 2, c)$, and Section 1 names the other limits of the search.

6.3 Boundary fibres

Proposition 6.3. *The formula for P_0 in Theorem F(i) holds for every content $(a, 2, c)$.*

Proof. If $x = 0$, then v_1 and v_2 lie on the vertical axis, with heights summing to $2c - y = m$. A path passes through $v_2 = (0, i)$ exactly when it starts with i north steps, and then also through $v_1 = (0, m - i)$; there are $\binom{a+c-i}{a}$ such paths, and $0 \leq m - i \leq i \leq c$ means $\lceil m/2 \rceil \leq i \leq \min(m, c)$. $\square$

Symmetry gives the other edges of Theorem F(i); also $f(j) = P_0(2j)$, $P_0(0) = P'_0(0) = \tau$.

Lemma 6.4. *For $0 \leq j < c$, $f(j+1) - f(j) = \binom{a+c-2j-1}{a} + \binom{a+c-2j-2}{a} - \binom{a+c-j}{a}$, and $\varrho(j) = [\binom{a+c-2j-1}{a} + \binom{a+c-2j-2}{a}] / \binom{a+c-j}{a}$ strictly decreases in j while it is positive, and is 0 afterwards. So f strictly increases while $\varrho(j) > 1$, has $f(j+1) = f(j)$ for at most one j , where $\varrho(j) = 1$, and strictly decreases afterwards; and $j^* = \min\{j : \varrho(j) \leq 1\}$.*

Proof. As $\binom{a+c-i}{a} = 0$ for $i > c$, $f(j) = \sum_{i=j}^{2j} \binom{a+c-i}{a}$, which gives the difference. It suffices that $\varrho_1(j) = \binom{a+c-2j-1}{a} / \binom{a+c-j}{a}$ and $\varrho_2(j) = \binom{a+c-2j-2}{a} / \binom{a+c-j}{a}$ strictly decrease while positive. If $\varrho_1(j+1) > 0$, then $p = a+c-2j-1 \geq a+2$, and with $q = a+c-j > p$, $\varrho_1(j+1)/\varrho_1(j) = \binom{p-2}{a} \binom{q}{a} / (\binom{p}{a} \binom{q-1}{a}) = \frac{p-a}{p} \cdot \frac{q}{q-a} \cdot \frac{p-a-1}{p-1} < 1$, as $(p-a)/p < (q-a)/q$; the same with $p = a+c-2j-2$ handles ϱ_2 . The minimum exists, as $\varrho(c-1) < 1$: its numerator vanishes if $c \geq 2$, and $\varrho(0) = 1/(a+1)$ if $c = 1$. $\square$

Corollary 6.5. (i) $f(1) - f(0) = \binom{a+c-2}{a} - \binom{a+c-1}{a-1}$, and $\binom{a+c-2}{a} / \binom{a+c-1}{a-1} = c(c-1)/(a(a+c-1))$ for $c \geq 2$; so $f(1) > \tau$ if and only if $c(c-1) > a(a+c-1)$ (for $c = 1$, $f(1) = 1 < \tau$). (ii) If $c(c-1) \leq a(a+c-1)$, then $f(j) < \tau$ for $j \geq 2$, and $f(1) < \tau$ unless $c(c-1) = a(a+c-1)$, when $f(1) = \tau$. (iii) $P_0(m) < \max_j f(j)$ for every odd m .

Proof. (i) is the difference at $j = 0$, with $\binom{a+c}{a} - \binom{a+c-1}{a} = \binom{a+c-1}{a-1}$. (ii) follows from Lemma 6.4, as $\varrho(0) \leq 1$ forces $\varrho(j) < 1$ for $j \geq 1$. (iii) If $2j \geq c$, then $f(j+1) = f(j) - \binom{a+c-j}{a} < f(j)$, and if $2j+1 = c$, then $f(j+1) - f(j) = 1 - \binom{a+c-j}{a} < 0$; so $f(j) > f(j-1)$ forces $2j \leq c$, and $f(j+1) = f(j)$ forces $2j+2 \leq c$. Next, $P_0(2j+1) \leq f(j+1) = P_0(2j+2)$, both being sums of the same positive terms, over $j+1 \leq i \leq \min(2j+1, c)$ and $j+1 \leq i \leq \min(2j+2, c)$, with equality only if $c \leq 2j+1$. Now $P_0(1) = \binom{a+c-1}{a} < \tau$, and for $m = 2j+1 \geq 3$, $P_0(m) = f(j^*)$ would need $c \leq 2j+1$ and either $j+1 = j^*$, so that $f(j^*) > f(j^*-1)$ and $2j+2 \leq c$, or $j = j^*$ and $f(j^*+1) = f(j^*)$, so that again $2j+2 \leq c$; neither is possible. $\square$

Proposition 6.6. *If $a \leq c$, then $\max_{m'} P'_0(m') = \tau$, attained only at $m' = 0$. So the largest boundary fibre of $(a, 2, c)$ has $f(j^*)$ words and occurs only on the edges $x = 0$ and $x = 2a$, at the even offsets $m = 2j$ with $f(j) = f(j^*)$: as the classes $(0, 2c-2j, 0)$ and $(2a, 2j, 2aj)$.*

Proof. For $(c, 2, a)$ the threshold of Corollary 6.5 compares $a(a-1)$ with $c(c+a-1)$, and $a(a-1) < a(2a-1) \leq c(c+a-1)$; so $P'_0(m') < \tau$ for even $m' \geq 2$ by Corollary 6.5(ii) and for odd m' by Corollary 6.5(iii). The rest is Corollary 6.5(iii) for $(a, 2, c)$ and Lemma 6.2(ii). $\square$

Remark 6.7. For fixed a , $j^*/c \rightarrow \zeta_a = (2^{1/a} - 1)/(2^{1+1/a} - 1)$ as $c \rightarrow \infty$, and $\zeta_a \sim (\ln 2)/a$. For $s \in (0, \frac{1}{2})$ and $j = \lfloor sc \rfloor$, $\varrho_1(j)$ and $\varrho_2(j)$ are products of a factors tending to $(1-2s)/(1-s)$, so $\varrho(j) \rightarrow 2((1-2s)/(1-s))^a$, which exceeds 1 exactly for $s < \zeta_a$; as ϱ decreases, $j^* > sc$ for large c if $s < \zeta_a$, and $j^* \leq sc$ if $s > \zeta_a$. We claim no asymptotics for $\beta(a, c)$.

6.4 Classes off and on the Cauchy surface

Lemma 6.8. *If $\Delta > 0$, then $\rho(x, y, z) \leq \tau$, with equality if and only if $(x, y, z) = (a, c, 0)$.*

Proof. By Lemma 6.1, $\rho(x, y, z)$ counts the triples (ω, v_1, v_2) with $v_1 + v_2 = 2(\mu, \nu)$ and $\delta\varepsilon = \Delta$, and a path carries at most one such pair: of two, say with $v_1 \leq v'_1$ (the points of a path are totally ordered), $v'_2 \leq v_2$, so $\delta' \leq \delta$ and $\varepsilon' \leq \varepsilon$, and $\delta'\varepsilon' = \delta\varepsilon > 0$ forces equality, whence $2(v'_1 - v_1) = (\delta - \delta', \varepsilon - \varepsilon') = 0$. So $\rho \leq \tau$. If every path carries a pair, the paths with all east steps first and with all north steps first carry pairs $(i, 0)$, (a, j) and $(0, j')$, (i', c) ; equal sums give $i+a = i' \leq a$ and $j = j' + c \leq c$, so $i = j' = 0$, $(\mu, \nu) = (a/2, c/2)$ and $\Delta = ac$, that is, $(x, y, z) = (a, c, 0)$. Conversely every path contains $(0, 0)$ and (a, c) , so $\rho(a, c, 0) = \tau$. $\square$

A class on the Cauchy surface can have more than τ words: the class $(x, y, z) = (24, 2, 24)$ of the content $(24, 2, 2)$ has $820 > \tau = 325$ words (`checks/check_r7_golden.py`, sections [3] and [7]). So it must be compared with the larger boundary fibres $P_0(m)$.

Lemma 6.9. *For every interior centre (μ, ν) , $K_{a,c}(\mu, \nu)$ is smaller than the largest boundary ψ_2 -fibre, except that $K_{2,2}(1, 1)$, the class $(x, y, z) = (2, 2, 2)$ of the content $(2, 2, 2)$, equals it.*

Appendix A proves Lemma 6.9: after a reduction by (6.2), both sides are expanded in the numbers of paths from one column to (a, c) and compared by two summations by parts; an exact finite certificate covers finitely many centres in the columns $\mu = 2, 3, 4$, one exceptional centre and the constants of an asymptotic estimate.

6.5 Proof of Theorem F

Lemma 6.10 (cf. [39, §2]). *For $1 \leq a \leq c$, $c(c-1) = a(a+c-1)$ if and only if $(a, c) = (F_n F_{n+1}, F_n F_{n+2})$ for an odd $n \geq 1$.*

Proof. The equation holds if and only if $c = (a+1+r)/2$ with an integer $r \geq 0$, $r^2 = 5a^2 - 2a + 1$ (then $r \equiv a+1 \pmod{2}$, so c is an integer), that is, $(5a-1)^2 - 5r^2 = -4$. So the solutions correspond, through $a = (u+1)/5$ and $c = (v+a+1)/2$, to the positive solutions (u, v) of $u^2 - 5v^2 = -4$ with $u \equiv 4 \pmod{5}$. If (u, v) is a positive solution with $v \geq 2$, then $u' = (3u-5v)/2$ and $v' = (3v-u)/2$ are integers ($u \equiv v \pmod{2}$), $(u' + v'\sqrt{5})/2$ is $(u + v\sqrt{5})/2$ times the unit $(3 - \sqrt{5})/2$, so $u'^2 - 5v'^2 = -4$, and $0 < v' < v$, $u' > 0$, as $u^2 = 5v^2 - 4 < 9v^2$, $9u^2 = 45v^2 - 36 > 25v^2$ and $u > v$. Descending, every positive solution comes from $(1, 1)$: $(u + v\sqrt{5})/2 = \varphi^{2s+1}$, $s \geq 0$; as $\varphi^k = (L_k + F_k\sqrt{5})/2$, the solutions are (L_k, F_k) with k odd. The Lucas numbers modulo 5 have period 4, starting 2, 1, 3, 4, so $L_k \equiv 4 \pmod{5}$ if and only if $k \equiv 3 \pmod{4}$, that is, $k = 2n+1$ with n odd. Then $(a, c) = (F_n F_{n+1}, F_n F_{n+2})$ is a solution: $c - a = F_n^2$, $a + c = F_n F_{n+3}$ and $c(c-1) - a(a+c-1) = (c-a)(a+c-1) - ac = F_n^2(F_n F_{n+3} - F_{n+1} F_{n+2} - 1) = 0$, by Cassini's identity $F_n F_{n+3} - F_{n+1} F_{n+2} = F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1} = 1$. Its $v = 2c - a - 1 = F_n F_{n+2} + F_n^2 - 1 = F_{n+1}^2 + F_n^2 = F_{2n+1}$ determines it, so it is the solution with $k = 2n+1$. $\square$

An independent search finds no other solution with $a \leq 10^6$ (`checks/check_r7_golden.py`, section [4]).

Proof of Theorem F. Part (i) is Proposition 6.3, the symmetries and Lemma 6.2(ii); part (ii) is Lemma 6.4. (iii) A class is a boundary fibre, or interior with $\Delta > 0$ and at most $\tau = P_0(0)$ words (Lemma 6.8), or interior with $\Delta = 0$ and smaller than the largest boundary fibre, with one exception (Lemma 6.9); so $\beta(a, c)$ is the largest boundary fibre, $f(j^*)$ by Proposition 6.6. (iv) By Lemma 6.4, $\beta(a, c) = \tau$ if and only if $f(1) \leq f(0)$, that is, $c(c-1) \leq a(a+c-1)$ (Corollary 6.5(i)); otherwise $\beta(a, c) = f(j^*) > \tau$. The roots of $c^2 - (a+1)c - (a^2 - a)$ have product $-(a^2 - a) \leq 0$, so for $c > 0$ the condition is $c \leq c^*(a)$. With $r = \sqrt{5a^2 - 2a + 1}$, $c^*(a) - \varphi a = (1 + r - \sqrt{5}a)/2$, where $r - \sqrt{5}a = -(2a-1)/(r + \sqrt{5}a)$ exceeds $-1/\sqrt{5}$ as $r > \sqrt{5}(a-1)$ ($8a > 4$), equals $2 - \sqrt{5}$ at $a = 1$, and is smaller for $a \geq 2$: $r \leq \sqrt{5}a$ gives $(\sqrt{5} - 2)(r + \sqrt{5}a) \leq (10 - 4\sqrt{5})a < 2a - 1$, as $(4\sqrt{5} - 8)a \geq 8\sqrt{5} - 16 > 1$. So $c^*(a) - \varphi a \in ((1 - 1/\sqrt{5})/2, (3 - \sqrt{5})/2]$, with the maximum at $a = 1$, where $c^*(1) = 2$. (v) By Lemmas 6.8 and 6.9, the only interior candidates are $(a, c, 0)$, when $\beta(a, c) = \tau$, and $(2, 2, 2)$ for $(a, c) = (2, 2)$; on the boundary (Proposition 6.6) the largest classes are $(0, 2c - 2j, 0)$ and $(2a, 2j, 2aj)$ with $f(j) = \beta(a, c)$. If $c(c-1) < a(a+c-1)$, only $j = 0$ qualifies (Corollary 6.5(ii)); if $c(c-1) = a(a+c-1)$, exactly $j = 0, 1$, and $(a, c) \neq (2, 2)$; if $c(c-1) > a(a+c-1)$, exactly $j = j^*$ and, if $f(j^* + 1) = f(j^*)$, $j^* + 1$, and no interior class qualifies. (vi) is Lemma 6.10 with the second case of (v), with five distinct classes. $\square$

6.6 The largest ψ_2 -fibre

Theorem G (The unique largest ψ_2 -fibre). *Let $a, c \geq 1$.*

- (i) $P(x, y) \leq P(a, c)$ for every (x, y) in the box. If $(a, c) \neq (1, 1)$, the inequality is strict for $(x, y) \neq (a, c)$: the centre (a, c) is the unique largest ψ_2 -fibre of $(a, 2, c)$. For $(a, c) = (1, 1)$ the fibres $(0, 2)$, $(1, 1)$ and $(2, 0)$ all have 2 words.
- (ii) (Andrews and Paule [1, eq. (2.2), p. 4].) With $m_0 = \lfloor (a + c)/2 \rfloor$,

$$P(a, c) = \sum_{p, q \geq 0} \binom{p+q}{p}^2 \binom{a+c-2p-2q}{a-2p} = (m_0 + 1) \binom{m_0}{\lfloor a/2 \rfloor} \binom{m_0}{\lceil a/2 \rceil}.$$

Case by case, the centre value is $(m_0 + 1) \binom{m_0}{a/2}^2$ if a and c are even, $(m_0 + 1) \binom{m_0}{(a-1)/2} \binom{m_0}{(a+1)/2}$ if both are odd, and $(m_0 + 1) \binom{m_0}{e/2} \binom{m_0+1}{e/2+1}$ if $a + c$ is odd and e is the even one of a, c . The first equality in (ii) is the lattice-path model at the centre; the second is [1, eq. (2.2)] with their $(m, n, i, j) = (c, a, q, p)$, a parity check matching their right side with our closed form. For $a = c$ even it is their (1.1) [1, p. 1], there credited to Problem E3376 of the American Mathematical Monthly (1990); see also [23]. Part (i) was not located; its proof, which also gives the closed form, uses neither the identity nor Theorem D.

Corollary 6.11. *For the content $(a, 2, a)$ with $a \geq 2$ put $D(x) = P(x, 2a - x)$. Then $D(x) < D(a) = (a + 1) \binom{a}{\lfloor a/2 \rfloor}^2$ for $x \neq a$, $0 \leq x \leq 2a$, and (a, a) is the unique largest ψ_2 -fibre.*

Proof. Take $c = a$ in Theorem G: $m_0 = a$ and $\binom{a}{\lfloor a/2 \rfloor} = \binom{a}{\lceil a/2 \rceil}$. $\square$

The function D is not monotone on either side of the centre: for $a = 4$ its values are 70, 40, 96, 88, 180, 88, 96, 40, 70, with $180 = 5 \cdot 6^2$, and for $a = 6$, $D(4) = 1620 > D(5) = 1608 < D(6) = 2800 = 7 \cdot 20^2$ (checks/check_h1_centre.py, section [7]); so the proof compares every fibre with the centre directly. For two letters, $|w|_{12}$ is distributed as the coefficients of a Gaussian binomial coefficient, whose central coefficient is a largest one (Sylvester [41, Theorem 11, p. 516]) but not always the only one: $\binom{5}{2}_q$ has the coefficients 1, 1, 2, 2, 2, 1, 1 (see [28, Theorem 6, p. 3]). Theorem G concerns two such statistics, $x = |w|_{12}$ and $y' = |w|_{32}$, which share the letters 2, and does not follow from this; in the Mann–Whitney reading of Remark 5.3, it puts the unique mode of their joint null distribution at their means (a, c) if $(a, c) \neq (1, 1)$, and the exact-distribution literature [43, 17, 46, 11], seen only as abstracts or metadata, limits this search. Theorem G does not extend to every b (Remark 5.4 for $b = 1$, Theorem D(iii) for odd b , and Proposition 5.5: for $(7, 6, 7)$ and $(5, 10, 5)$ the centre loses to off-centre fibres); whether it extends to $b \equiv 0 \pmod{4}$ is part of Question 5.6.

6.7 Proof of Theorem G

The proof folds a word, a ± 1 walk with two marked times, at their midpoint into an occupation count of a planar simple walk (Lemma 6.14), and compares occupation counts through an exchangeability count (Lemma 6.15) and the elementary fact, not claimed, that a symmetric walk spends the most time at its start (Lemma 6.16).

For $t \geq 0$ and $j \in \mathbb{Z}$ let $B_t(j) = \binom{t}{(t+j)/2}$ if $|j| \leq t$ and $j \equiv t \pmod{2}$, and $B_t(j) = 0$ otherwise: the number of ± 1 walks of length t from 0 to j . So $B_t(-j) = B_t(j)$, $\sum_p B_t(p) B_{t'}(j-p) = B_{t+t'}(j)$ and $B_{t+1}(j) = B_t(j-1) + B_t(j+1)$. Let $\mathcal{V}_m$ be the set of the 4^m planar simple walks $V = (X, Y)$ of length m from $(0, 0)$, with steps $(\pm 1, 0)$ and $(0, \pm 1)$. The rotation $(\omega_1, \omega_2) \mapsto V = ((\omega_1 + \omega_2)/2, (\omega_1 - \omega_2)/2)$ is a bijection from the pairs of ± 1 walks of length m onto $\mathcal{V}_m$: the step pairs $(+, +), (-, -), (+, -), (-, +)$ give $(1, 0), (-1, 0), (0, 1), (0, -1)$. Let

$\text{Occ}_m(j, k) = \#\{(V, t) : V \in \mathcal{V}_m, 0 \leq t \leq m, X(t) = j, Y(m) = k\}$ be the total time that the walks ending at height k spend at the level $X = j$, and $N_h(j) = \sum_{i=0}^h 2^{h-i} B_i(j)$ the number of pairs of a ± 1 walk of length h and a time $i \leq h$ at which it is at j . In this subsection $\kappa = c - a$, and $\sigma = x + y'$ and $\chi = y' - x$ for (x, y) in the box, where $y' = 2c - y$.

Lemma 6.12. $\#\{V \in \mathcal{V}_t : Y(t) = j\} = B_{2t}(2j)$.

Proof. By the rotation it is $\#\{(\omega_1, \omega_2) : \omega_1(t) - \omega_2(t) = 2j\} = \sum_p B_t(p) B_t(p - 2j)$, which is $B_{2t}(2j)$ by symmetry and splitting. $\square$

Lemma 6.13. Let Ω be the set of ± 1 walks ω of length $a + c$ with c up-steps, so $\omega(a + c) = \kappa$. Then $P(x, y)$ counts the triples of a walk $\omega \in \Omega$ and times $s_1 \leq s_2$ with $s_1 + s_2 = \sigma$ and $\omega(s_1) + \omega(s_2) = \chi$.

Proof. Read the $\{1, 3\}$ -subword as ω , with 3 up and 1 down (the path of Lemma 6.1 turned by 45°), and let s_l be the number of letters 1 and 3 before the l -th letter 2; this is a bijection onto $\Omega \times \{s_1 \leq s_2\}$. Of the first s letters, $(s - \omega(s))/2$ are 1s and $(s + \omega(s))/2$ are 3s, so $x = \sum_l (s_l - \omega(s_l))/2$ and $y' = |w|_{32} = \sum_l (s_l + \omega(s_l))/2$. $\square$

Lemma 6.13 restates Lemma 2.4 and is not claimed. Reversal maps σ to $2(a + c) - \sigma$ and fixes the centre, where $\sigma = a + c$; the mirror map sends $(a, c; x, y)$ to $(c, a; y, x)$.

Lemma 6.14. Let $\sigma = 2m + \iota \leq a + c$ with $\iota \in \{0, 1\}$, and let $I_0 = \{0\}$, $I_1 = \{1, -1\}$. Then

$$P(x, y) = \sum_{\theta \in I_\iota} \sum_{k \in \mathbb{Z}} \text{Occ}_m(k + \frac{\chi - \theta}{2}, k) B_{a+c-2m-\iota}(\kappa - \theta + 2k), \quad (6.3)$$

where $(\chi - \theta)/2 \in \mathbb{Z}$, since $\chi \equiv \sigma \equiv \theta \pmod{2}$.

Proof. In Lemma 6.13 write $s_1 = m - r$ and $s_2 = m + \iota + r$, where $0 \leq r \leq m$ as $s_1 \geq 0$ and $s_2 \leq a + c$. Cut ω bijectively into $\theta = \omega(m + \iota) - \omega(m) \in I_\iota$, the reversed first part $\omega^-(t) = \omega(m - t) - \omega(m)$, $0 \leq t \leq m$, and the second part $\omega^+(t) = \omega(m + \iota + t) - \omega(m + \iota)$. As $\omega(m) = -\omega^-(m)$, the two conditions become: ω^+ has increment $\kappa - \theta + \omega^-(m)$, and $\omega^-(r) + \omega^+(r) = \chi - \theta + 2\omega^-(m)$. Split ω^+ into its first m steps ω_2 and a walk ω'' of length $a + c - 2m - \iota$, and rotate (ω^-, ω_2) to $V = (X, Y) \in \mathcal{V}_m$. As $\omega^- + \omega_2 = 2X$ and $\omega^- - \omega_2 = 2Y$, the conditions become $X(r) - X(m) = Y(m) + (\chi - \theta)/2$ and that ω'' has increment $\kappa - \theta + 2Y(m)$. Finally, $\hat{V}$ with $\hat{X}(t) = X(m - t) - X(m)$ and $\hat{Y}(t) = Y(m) - Y(m - t)$ has the steps of V in reverse order, the horizontal ones negated, so $V \mapsto \hat{V}$ is an involution of $\mathcal{V}_m$; as $\hat{X}(m - r) = X(r) - X(m)$ and $\hat{Y}(m) = Y(m)$, the first condition reads $\hat{X}(t) = \hat{Y}(m) + (\chi - \theta)/2$ with $t = m - r$. For fixed θ and $k = \hat{Y}(m)$ the pairs $(\hat{V}, t)$ number $\text{Occ}_m(k + (\chi - \theta)/2, k)$, and the walks ω'' number $B_{a+c-2m-\iota}(\kappa - \theta + 2k)$. $\square$

At the centre, $\sigma = a + c$ and $\chi = \kappa$, and B_0 selects the single k with level 0:

$$P(a, c) = \begin{cases} \text{Occ}_{(a+c)/2}(0, -\kappa/2) & \text{if } a + c \text{ is even,} \\ \sum_{\theta=\pm 1} \text{Occ}_{(a+c-1)/2}(0, (\theta - \kappa)/2) & \text{if } a + c \text{ is odd.} \end{cases} \quad (6.4)$$

Lemma 6.15. For $m \geq 0$ and $j, k \in \mathbb{Z}$,

$$\text{Occ}_m(j, k) = (m + 1) \sum_{h=0}^m \frac{1}{h + 1} \binom{m}{h} B_{m-h}(k) N_h(j). \quad (6.5)$$

Proof. Classify $V \in \mathcal{V}_m$ by its type $\mathbf{t} \in \{\mathbf{h}, \mathbf{v}\}^m$, the sequence of horizontal and vertical steps. Given $\mathbf{t}$ with h letters $\mathbf{h}$, V is given by its horizontal steps, a ± 1 walk $\omega^{\mathbf{h}}$ of length h , and its vertical steps, with sum $Y(m)$; and $X(t) = \omega^{\mathbf{h}}(i_{\mathbf{t}}(t))$, where $i_{\mathbf{t}}(t)$ is the number of letters $\mathbf{h}$ among the first t letters of $\mathbf{t}$. So, with $o_i(\mathbf{t}) = \#\{t \in [0, m] : i_{\mathbf{t}}(t) = i\}$, $\text{Occ}_m(j, k) = \sum_{\mathbf{t}} B_{m-h}(k) \sum_{i=0}^h o_i(\mathbf{t}) 2^{h-i} B_i(j)$. Now $o_i(\mathbf{t}) - 1$ is the number of letters $\mathbf{v}$ between the i -th and $(i+1)$ -th letter $\mathbf{h}$ (before the first, or after the last), and $\mathbf{t} \mapsto (o_i(\mathbf{t}) - 1)_{i=0}^h$ is a bijection onto the weak compositions of $m - h$ into $h + 1$ parts, a set invariant under permuting the parts. So $\sum_{\mathbf{t}} o_i(\mathbf{t})$ does not depend on i ; as $\sum_i o_i(\mathbf{t}) = m + 1$, it is $\binom{m}{h}(m + 1)/(h + 1)$. $\square$

Lemma 6.16. For $h \geq 0$ and $j \neq 0$, $N_h(-j) = N_h(j) < N_h(0)$.

Proof. Symmetry is $B_i(-j) = B_i(j)$. For even j , $B_i(j) \leq B_i(0)$ for every i , and $B_0(j) = 0 < 1 = B_0(0)$. For odd j , $B_i(j) = 0$ for even i , and for odd $i = 2l + 1 \leq h$, $2^{h-2l-1} B_{2l+1}(j) \leq 2^{h-2l-1} \binom{2l+1}{l} = 2^{h-2l} \binom{2l}{l} \frac{2l+1}{2l+2} < 2^{h-2l} B_{2l}(0)$; matching $2l + 1$ with $2l$ gives $N_h(j) < N_h(0)$ for $h \geq 1$, and $N_0(j) = 0 < 1 = N_0(0)$. $\square$

Corollary 6.17. $\text{Occ}_m(0, -k) = \text{Occ}_m(0, k)$, and $\text{Occ}_m(j, k) \leq \text{Occ}_m(0, k)$, strictly if and only if $j \neq 0$ and $|k| \leq m$.

Proof. By (6.5), $\text{Occ}_m(0, k) - \text{Occ}_m(j, k) = (m + 1) \sum_h \binom{m}{h} B_{m-h}(k) (N_h(0) - N_h(j))/(h + 1)$, a sum of nonnegative terms by Lemma 6.16; for $j \neq 0$ and $|k| \leq m$ the term $h = m - |k|$ is positive, as $B_{|k|}(k) = 1$; for $j = 0$ it vanishes, and for $|k| > m$ every $B_{m-h}(k)$ vanishes. Symmetry is $B_{m-h}(-k) = B_{m-h}(k)$. $\square$

Proof of Theorem G(i). Let $m_0 = \lfloor (a + c)/2 \rfloor$; then $|\kappa| \leq a + c - 2$. Let $\mathcal{J} = \{\kappa/2\}$ if $a + c$ is even and $\mathcal{J} = \{(\kappa \pm 1)/2\}$ if $a + c$ is odd; then $|j_0| \leq m_0$ for $j_0 \in \mathcal{J}$. For $0 \leq m \leq m_0$ let $\Xi(m) = \#\{(V, t) : V \in \mathcal{V}_{m_0}, 0 \leq t \leq m, X(t) = 0, Y(m_0) \in \mathcal{J}\}$, nondecreasing in m .

Step 1. $P(a, c) = \Xi(m_0)$, by (6.4) and the symmetry in Corollary 6.17.

Step 2. If $\sigma = 2m + \iota \leq a + c$, then

$$P(x, y) \leq \sum_k \text{Occ}_m(0, k) B_{a+c-2m}(\kappa + 2k) = \Xi(m). \quad (6.6)$$

The inequality comes from (6.3) and Corollary 6.17, with $B_t(\kappa - 1 + 2k) + B_t(\kappa + 1 + 2k) = B_{t+1}(\kappa + 2k)$ for $\iota = 1$. For the equality, split $V \in \mathcal{V}_{m_0}$ after m steps and keep $t \leq m$: by Lemma 6.12 and $k \mapsto -k$, $\Xi(m) = \sum_k \text{Occ}_m(0, k) \sum_{j_0 \in \mathcal{J}} B_{2(m_0-m)}(2j_0 + 2k)$, and the inner sum is $B_{a+c-2m}(\kappa + 2k)$ in both parities. So $P(x, y) \leq \Xi(m) \leq \Xi(m_0) = P(a, c)$ if $\sigma \leq a + c$, and by reversal also if $\sigma > a + c$.

Step 3. Let $(a, c) \neq (1, 1)$, so $a + c \geq 3$; by the mirror map let $a \leq c$, so $0 \leq \kappa \leq a + c - 2$; and let $(x, y) \neq (a, c)$ with, by reversal, $\sigma = 2m + \iota \leq a + c$. As (σ, χ) determines (x, y) , $\sigma = a + c$ forces $\chi \neq \kappa$. Two observations settle every case.

(S1) $\Xi(m) < \Xi(m_0)$ if $m \leq m_0 - 2$, and if $m = m_0 - 1$ unless $a + c$ is even and a odd. We need $V \in \mathcal{V}_{m_0}$ and $t \in (m, m_0]$ with $X(t) = 0$ and $Y(m_0) = j_0 \in \mathcal{J}$. For $t = m + 1 \leq m_0 - 1$, take p vertical steps towards j_0 (either way if $j_0 = 0$), where $p = t$ if $|j_0| \geq t$ and otherwise $p \in \{|j_0|, |j_0| + 1\}$ has the parity of t , then $(t - p)/2$ east-west pairs: $X(t) = 0$ and $|j_0 - Y(t)| \leq \max(|j_0| - t, 1) \leq m_0 - t$, so the remaining steps can reach j_0 . For $t = m_0$: $X(m_0) = 0$ forces $Y(m_0) \equiv m_0 \pmod{2}$, and conversely $|j_0|$ vertical steps and $(m_0 - |j_0|)/2$ east-west pairs work if $j_0 \equiv m_0$; for $a + c$ odd one element of $\mathcal{J}$ has this parity, and for $a + c$ even, $\kappa/2 \equiv m_0$ if and only if $m_0 - \kappa/2 = a$ is even.

(S2) The inequality in (6.6) is strict if (6.3) has a term with $B_{a+c-2m-\iota}(\kappa - \theta + 2k) > 0$, $|k| \leq m$ and level $k + (\chi - \theta)/2 \neq 0$ (Corollary 6.17).

For $a + c$ odd only $m = m_0$, that is $\sigma \in \{a + c - 1, a + c\}$, escapes (S1); for $a + c$ even only $\sigma = a + c$ and, when a is odd, $\sigma \in \{a + c - 2, a + c - 1\}$. These cases are settled by (S2):

(1) $\sigma = a + c$ even: the weight at $k = -\kappa/2$ has $|k| \leq m_0$ and level $(\chi - \kappa)/2 \neq 0$.

- (2) $\sigma = a + c$ odd: for $\theta = 1$ the weight at $k = (1 - \kappa)/2$ has $|k| \leq m_0$ and level $(\chi - \kappa)/2 \neq 0$.
- (3) $\sigma = a + c - 1$, $a + c$ odd ($\iota = 0$, $m = m_0$): the weights $B_1(\kappa + 2k)$ sit at $k = (-\kappa \pm 1)/2$, with $|k| \leq m_0$ and levels differing by 1.
- (4) $\sigma = a + c - 1$, $a + c$ even, a odd ($\iota = 1$, $m = m_0 - 1$): for $\theta = 1$ the weights sit at $k = -\kappa/2, 1 - \kappa/2$, with $|k| \leq m_0 - 1$ ($\kappa \leq a + c - 2$, $a + c \geq 4$) and levels differing by 1.
- (5) $\sigma = a + c - 2$, $a + c$ even, a odd ($\iota = 0$, $m = m_0 - 1$): the weights $B_2(\kappa + 2k)$ include $k = -\kappa/2, 1 - \kappa/2$, with $|k| \leq m_0 - 1$ and levels differing by 1.
- Finally, for $(a, c) = (1, 1)$, $P(0, 2) = P(2, 0) = \tau = 2$ by Proposition 6.3 and reversal, and $P(1, 1) = 2$, the words 2132 and 2312. $\square$

The closed form in (ii) needs two more counts.

Lemma 6.18. $N_h(0) = (h + 1) \binom{h}{\lfloor h/2 \rfloor}$ for $h \geq 0$.

Proof. $N_{2l}(0) = \sum_{i=0}^l 4^{l-i} \binom{2i}{i}$ and $N_{2l+1}(0) = 2N_{2l}(0)$. Now $N_0(0) = 1$, $N_{2l}(0) = 4N_{2l-2}(0) + \binom{2l}{l}$, and $(2l + 1) \binom{2l}{l}$ satisfies the same recursion, because $\binom{2l}{l} = \binom{2l-2}{l-1} \cdot 2(2l - 1)/l$ gives $4(2l - 1) \binom{2l-2}{l-1} + \binom{2l}{l} = (2l + 1) \binom{2l}{l}$. So $N_{2l}(0) = (2l + 1) \binom{2l}{l}$ and $N_{2l+1}(0) = (2l + 2) \binom{2l+1}{l+1}$. $\square$

Lemma 6.19. For $|k| \leq m$, $\text{Occ}_m(0, k) = (m + 1)B_m(k)^2$ if $m - k$ is even, and $\text{Occ}_m(0, k) = (m + 1)B_m(k + 1)B_m(k - 1)$ if $m - k$ is odd.

Proof. Let $\iota \in \{0, 1\}$, $\iota \equiv m - k \pmod{2}$. By (6.5) and Lemma 6.18, $\text{Occ}_m(0, k) = (m + 1) \sum_h \binom{m}{h} B_{m-h}(k) \binom{h}{\lfloor h/2 \rfloor}$; a term is nonzero only if $h \equiv \iota$, and then $\binom{h}{\lfloor h/2 \rfloor} = B_h(\iota)$. So the sum counts the $V \in \mathcal{V}_m$ with $X(m) = \iota$ and $Y(m) = k$ (choose the positions of the horizontal steps, then the values of all steps), which by the rotation number $B_m(k + \iota)B_m(k - \iota)$. $\square$

Proof of Theorem G(ii). At the centre $v_1 + v_2 = (a, c)$; with $v_1 = (p, q)$, a path runs from $(0, 0)$ to v_1 , from v_1 to $v_2 = (a - p, c - q)$ and from v_2 to (a, c) in $\binom{p+q}{p}$, $\binom{a+c-2p-2q}{a-2p}$ and $\binom{p+q}{p}$ ways. For the closed form, Step 1 gives $P(a, c) = \sum_{j_0 \in \mathcal{J}} \text{Occ}_{m_0}(0, j_0)$, and Lemma 6.19 applies, with $B_{m_0}(j) = \binom{m_0}{(m_0+j)/2}$. If $a + c$ is even, then $j_0 = \kappa/2$, $m_0 - j_0 = a$, and $P(a, c)$ is $(m_0 + 1)B_{m_0}(\kappa/2)^2 = (m_0 + 1) \binom{m_0}{a/2}^2$ for a, c even and $(m_0 + 1)B_{m_0}(\kappa/2 + 1)B_{m_0}(\kappa/2 - 1) = (m_0 + 1) \binom{m_0}{(a-1)/2} \binom{m_0}{(a+1)/2}$ for a, c odd. If $a + c$ is odd, then $m_0 - (\kappa - 1)/2 = a$; for a even the two terms are $(m_0 + 1) \binom{m_0}{(c-1)/2}^2$ and $(m_0 + 1) \binom{m_0}{(c+1)/2} \binom{m_0}{(c-1)/2}$, with sum $(m_0 + 1) \binom{m_0}{(c-1)/2} \binom{m_0+1}{(c+1)/2} = (m_0 + 1) \binom{m_0}{a/2} \binom{m_0+1}{a/2}$, and for a odd likewise $(m_0 + 1) \binom{m_0}{c/2} \binom{m_0+1}{c/2}$. By $\binom{n}{i} = \binom{n}{n-i}$, each case is $(m_0 + 1) \binom{m_0}{\lfloor a/2 \rfloor} \binom{a+c-m_0}{\lceil a/2 \rceil}$, and $a + c - m_0 = \lceil (a + c)/2 \rceil$. $\square$

7 Uniform contents

At the content (n^k) the alternating classes are whole ψ_2 -fibres and single trace classes, for $k \geq 4$ the only largest trace classes (Remark 7.1); for $n = 1$ they have the classical maximum of E_k words. A larger Parikh class must merge trace classes. An exact integral formula, under which the relative sizes obey a parity sandwich, shows that merging wins for $n \geq 4$, $k \geq 3$ and for $n = 3$, $k \geq 4$, while for $n = 2$ every one-end relaxation loses; computation suggests that the alternating classes of (2^k) are the only largest ones once $k \geq 6$.

Remark 7.1 (Largest trace classes). For $k \geq 4$ and every content $\mathbf{n}$, $Z(\mathbf{n})$ and $Z'(\mathbf{n})$ are the only largest trace classes, a direct consequence of Iriarte's theorem [15, Theorem 3], [14, Theorem 3.7] that the acyclic orientations of a connected comparability graph with the most linear extensions are exactly its transitive ones. Join two letter occurrences if their letters differ by at most 1; a word orients this connected graph from earlier to later occurrences, with chains on the cliques of equal letters, and its trace class is the set of linear extensions

(Lemma 7.4). Orienting the edges between cliques from the odd letter to the even one, or the reverse, is transitive and gives $Z(\mathbf{n})$, or $Z'(\mathbf{n})$; no other transitive orientation has chains on the cliques. (a) An occurrence y of an interior letter i is a source or a sink towards the neighbouring cliques: if $x \rightarrow y \rightarrow z$ with x, z in them, transitivity joins x and z , so they lie in one clique, and an occurrence z' of the other neighbour gives $x \rightarrow y \rightarrow z'$ or $z' \rightarrow y \rightarrow z$, which is impossible. (b) On the chain of i , a sink before a source would close a cycle through a neighbouring occurrence. (c) A source before a sink would give every occurrence of a neighbouring letter an in-edge and an out-edge to that chain, against (a) for an interior neighbour, which exists as $k \geq 4$. So the interior letters are all sources or all sinks, alternately. For an isolated content ($\min(n_i, n_{i+1}) = 1$ for every i) each area determines its projection (Lemma 2.6), so fibres, trace classes and Parikh classes coincide, and $Z(\mathbf{n})$, $Z'(\mathbf{n})$ are the only largest Parikh classes: for (1^k) this is the theorem of Niven and de Bruijn [42, Corollary 1.6.5], and for $\{1, 2\}$ -contents without two adjacent 2s the ‘if’ direction of Conjecture 7.30.

Setting. Throughout, the content is (n^k) , $n \geq 1$, $k \geq 2$, so $|\mathbf{n}| = nk$, and Z_k has $\psi_2 = (n^2, 0, n^2, 0, \dots)$. The edge-1 projections form $\mathcal{A}_1 = \bigcup_u \mathcal{A}_1(u)$, the nondecreasing $A \in \{0, \dots, n\}^n$ (equivalently, the words of content (n, n)); A is *non-full* if $A \neq (n^n)$, the projection of Z_k , that is, if $A_1 \leq n - 1$. For $\mathcal{S} \subseteq \mathcal{A}_1$, $Z_k(\mathcal{S})$ is the set of words with edge-1 projection in $\mathcal{S}$ and the projections of Z_k on the other edges, and $r_k(\mathcal{S}) = |Z_k(\mathcal{S})|/|Z_k|$ if $\mathcal{S}$ is a nonempty set of non-full projections. Let $\mathcal{S}_u = \mathcal{A}_1(u)$, of *edge-1 deficit* $n^2 - u$; $Z_k(\mathcal{S}_u)$ is the ψ_2 -fibre of $(u, 0, n^2, 0, \dots)$, as extreme edges have single projections, and one Parikh class (Corollary 3.3). A *one-end (two-end) relaxation* of $X \in \{Z_k, Z'_k\}$, $k \geq 3$, is the ψ_2 -fibre of a vector that differs from $\psi_2(X)$ exactly at one end edge (at both); a one-end relaxation is one Parikh class (Corollary 3.3) of the size of some $Z_k(\mathcal{S}_u)$ (Corollary 7.15).

Definition 7.2. Let $n = 2$. The projections of area 2 on edge i are $\mathbf{P} = i(i+1)(i+1)i = (1, 1)$ and $\mathbf{Q} = (i+1)ii(i+1) = (0, 2)$. A word w of the *central fibre* $\psi_2^{-1}(2, \dots, 2)$ has the *label word* $\mathbf{s}(w) = \mathbf{s}_1 \cdots \mathbf{s}_{k-1} \in \{\mathbf{P}, \mathbf{Q}\}^{k-1}$ of its projections and the *QP-descent set* $\mathbf{QP}(w) = \{i \in [k-2] : \mathbf{s}_i \mathbf{s}_{i+1} = \mathbf{QP}\}$. For $J \subseteq [k-2]$ let $\Theta_J = \{w \in \psi_2^{-1}(2, \dots, 2) : \mathbf{QP}(w) = J\}$.

Theorem H (Uniform contents). (i) *Let $n = 2$ and $k \geq 2$. For every w in the central fibre of (2^k) and $1 \leq j < l \leq k$ with $l - j \geq 2$, $|w|_{j..l} = 0$ if $\mathbf{QP}(w)$ meets $\{j, \dots, l-2\}$, and $|w|_{j..l} = 2$ otherwise. The Parikh classes of the fibre are exactly the sets Θ_J with $J \subseteq [k-2]$ without two consecutive integers; each is nonempty, so there are F_k of them.*

(ii) *Let $n \geq 1$ and let $\mathcal{S}$ be a nonempty set of non-full edge-1 projections. For all $i \geq 1$ and $k \geq 2i$, $r_{2i+1}(\mathcal{S}) \leq r_k(\mathcal{S}) \leq r_{2i}(\mathcal{S})$, with equality on the left only for $k = 2i + 1$ and on the right only for $k = 2i$. Hence*

$$r_3(\mathcal{S}) < r_5(\mathcal{S}) < r_7(\mathcal{S}) < \cdots < r_6(\mathcal{S}) < r_4(\mathcal{S}) < r_2(\mathcal{S}),$$

the limits $r_\infty^{\text{odd}}(\mathcal{S}) = \lim_i r_{2i+1}(\mathcal{S})$ and $r_\infty^{\text{even}}(\mathcal{S}) = \lim_i r_{2i}(\mathcal{S})$ exist, and $r_{2i+1}(\mathcal{S}) < r_\infty^{\text{odd}}(\mathcal{S}) \leq r_\infty^{\text{even}}(\mathcal{S}) < r_{2j}(\mathcal{S})$ for all $i, j \geq 1$.

(iii) *If $n \geq 4$ and $k \geq 3$, or if $n = 3$ and $k \geq 4$, no alternating class of (n^k) is a largest Parikh class. If $n = 2$ and $k \geq 3$, every one-end relaxation of an alternating class has at most $\frac{50}{53}|Z_k|$ words, with equality only for $k = 4$ and area 2 at the relaxed edge.*

Part (i) is a direct consequence of Simpson’s results on palindromes [38] (see after its proof); only the explicit QP description and the count were not located. For $k = 3$ it gives the values $\{0, 2\}$ of Proposition 4.13 at $b = 2$. For $n = 1$, (ii) is a direct consequence of [5] (Remark 7.16); its method (credited after Lemmas 7.8 and 7.14) is known and not claimed. Part (ii) for $n \geq 2$, part (iii) and Proposition 7.3 were not located; the literature on descent sets of multiset permutations, where analogues are likeliest, was not searched systematically.

7.1 The central fibre of (2^k) , and $(2, 2, 1)$ ends

Proof of Theorem H(i). In (2.1), $\mathbf{B}_t(w) = \mathbf{u}_t \mathbf{v}_t^\top$, where $(\mathbf{u}_t, \mathbf{v}_t)$ is $(\mathbf{e}_1, \mathbf{1})$ if $\mathbf{s}_t = \mathbf{P}$ and $(\mathbf{1}, \mathbf{e}_2)$ if $\mathbf{s}_t = \mathbf{Q}$, for the unit vectors $\mathbf{e}_1, \mathbf{e}_2$. So $|w|_{j..l} = (\mathbf{1}^\top \mathbf{u}_j) \prod_{t=j}^{l-2} (\mathbf{v}_t^\top \mathbf{u}_{t+1}) \cdot (\mathbf{v}_{l-1}^\top \mathbf{1})$, where $(\mathbf{1}^\top \mathbf{u}_t, \mathbf{v}_t^\top \mathbf{1})$ is $(1, 2)$ if $\mathbf{s}_t = \mathbf{P}$ and $(2, 1)$ if $\mathbf{s}_t = \mathbf{Q}$, and $\mathbf{v}_t^\top \mathbf{u}_{t+1}$ is $1, 2, 0, 1$ for $\mathbf{s}_t \mathbf{s}_{t+1} = \mathbf{PP}, \mathbf{PQ}, \mathbf{QP}, \mathbf{QQ}$. If $\mathbf{QP}(w)$ meets $\{j, \dots, l-2\}$, a factor vanishes. Otherwise $\mathbf{s}_j \dots \mathbf{s}_{l-1} = \mathbf{P}^i \mathbf{Q}^{l-j-i}$, and the product is 2: its only factor 2 is $\mathbf{1}^\top \mathbf{u}_j$ if $i = 0$, $\mathbf{v}_{l-1}^\top \mathbf{1}$ if $i = l-j$, and the factor of the step $\mathbf{PQ}$ otherwise. So the higher entries are functions of $\mathbf{QP}(w)$, and $t \in \mathbf{QP}(w)$ if and only if $|w|_{t..t+2} = 0$; the classes of the fibre are the nonempty sets Θ_J . No $\mathbf{QP}(w)$ contains t and $t+1$, which would force $\mathbf{s}_{t+1} = \mathbf{P}$ and $\mathbf{s}_{t+1} = \mathbf{Q}$. Conversely, if J has no two consecutive integers, the label word with $\mathbf{s}_t = \mathbf{Q}$ exactly for $t \in J$ has $\mathbf{QP}$ -descent set J and is realised by a word (Lemma 2.2). Such subsets of $[r]$ number F_{r+2} (one for $r = 0$, two for $r = 1$, and the Fibonacci recursion, according as r lies in the subset), hence F_k for $[k-2]$. $\square$

Simpson calls a palindrome in which every letter occurs twice *special*, and shows that the entries $|w|_{j..l}$, $l-j \geq 2$, of a special word are 0 or 2, that for $l-j \geq 3$ such an entry is 2 if and only if $|w|_{j..l-1} = |w|_{j+1..l} = 2$, and that the entries $|w|_{j..j+2}$ determine the matrix [38, Theorem 3(c),(d) and Corollary 4, p. 331]. The palindromes of content (2^k) lie in the central fibre, as $|w^R|_{i(i+1)} = 4 - x_i$, and each trace class of the fibre contains one, $u u^R$ for a permutation u of $[k]$ in which i precedes $i+1$ exactly when $\mathbf{s}_i = \mathbf{P}$. So Simpson's statements hold on the whole fibre, and with the case $l = j+2$ above they give the entries and the classes Θ_J of (i). The golden ratio in the growth of F_k is unrelated to that of Theorem F.

Proposition 7.3. *Let $k \geq 4$ and $\mathbf{n} = (2, 2, 1, n_4, \dots, n_k)$, with any $n_4, \dots, n_k \geq 1$. Let $\mathbf{x} = \psi_2(Z(\mathbf{n})) = (4, 0, n_4, 0, n_5 n_6, 0, \dots)$, and let $\mathbf{x}^*$ be $\mathbf{x}$ with x_1 replaced by 2. Then $\psi_2^{-1}(\mathbf{x}^*)$ is a single Parikh class with more words than $Z(\mathbf{n})$, so no alternating class is a largest Parikh class of $\mathbf{n}$. The same holds for every content $(n_1, \dots, n_{k-3}, 1, 2, 2)$ with $k \geq 4$.*

Proof. Edge 1 is the only non-extreme edge of $\mathbf{x}^*$, so $Z^* = \psi_2^{-1}(\mathbf{x}^*)$ is one Parikh class (Corollary 3.3). The words of $Z(\mathbf{n})$ and Z^* have $x_2 = 0$ and the projections of $Z(\mathbf{n})$ on the edges $3, \dots, k-1$: they are the shuffles, with the 3 before both 2s, of a word $\beta \in \mathcal{W}(2, 2)$ on $\{1, 2\}$ and a word v in the set $\mathcal{V}$ of words of content $(1, n_4, \dots, n_k)$ over $\{3, \dots, k\}$ with these projections. As $|w|_{12}$ is $0, 1, 2, 2, 3, 4$ for $\beta = 2211, 2121, 1221, 2112, 1212, 1122$, the words of $Z(\mathbf{n})$ have $\beta = 1122$ and those of Z^* have $\beta \in \{1221, 2112\}$. Fix v , with its 3 in position m , and let $N' = |v| = 1 + n_4 + \dots + n_k \geq 2$. The shuffles with exactly i letters of β before the 3 number $\binom{m-1+i}{i} \binom{N'-m+4-i}{4-i}$, and they are admissible if and only if i is less than the position 3, 2, 1 of the first 2 in 1122, 1221, 2112. So Z^* takes the terms $i = 0, 0, 1$ and $Z(\mathbf{n})$ the terms $i = 0, 1, 2$:

$$|Z^*| - |Z(\mathbf{n})| = \sum_{v \in \mathcal{V}} f(m), \quad f(m) = \binom{N' - m + 4}{4} - \binom{m + 1}{2} \binom{N' - m + 2}{2}.$$

In $\mathcal{V}$ the only condition on the 3 is that it precedes every 4, as edge 3 is full; so deleting it is a bijection onto the pairs (v'', m) with $1 \leq m \leq \ell(v'')$, where v'' runs over the analogous set $\mathcal{V}''$ of words over $\{4, \dots, k\}$, nonempty by Lemma 2.2 ($\mathcal{V}'' = \{4^{n_4}\}$ if $k = 4$), and $\ell(v'') \leq N' - 1$ is the position of the first 4 in v'' . It suffices to show that $\sum_{m \leq \ell} f(m) > 0$ for $1 \leq \ell \leq N' - 1$. For $1 \leq m \leq N'$, $f(m) / \binom{N'-m+2}{2} = (N' - m + 4)(N' - m + 3)/12 - m(m + 1)/2$ strictly decreases, $f(1) = \binom{N'+1}{2}((N' + 3)(N' + 2)/12 - 1) > 0$ and $f(N') = 1 - \binom{N'+1}{2} < 0$. So the partial sums $\sum_{m \leq \ell} f(m)$ increase from $f(1) > 0$ while $f > 0$, then do not increase, and end at $\sum_{m=1}^{N'} f(m) = 0$ (both parts summing to $\binom{N'+4}{5}$, by the hockey-stick identity and by Chu–Vandermonde) with the step $f(N') < 0$; hence they are positive for $\ell \leq N' - 1$. So $|Z^*| > |Z(\mathbf{n})| = |Z'(\mathbf{n})|$; the mirror statement follows (Lemma 2.3). $\square$

Over the 120 tails with $k \leq 7$ and $n_4, \dots, n_k \leq 3$, the least ratio $|Z^*|/|Z(\mathbf{n})|$ is 371/357, that is 53/51, at $\mathbf{n} = (2, 2, 1, 1, 3)$ (section [4] of `checks/check_r7_switch.py`).

7.2 An exact transfer

For a relation R on a finite set E , e_R is the number of bijections $\sigma : E \rightarrow [|E|]$ with $\sigma(x) < \sigma(y)$ for every $(x, y) \in R$; for a poset P , $e(P)$ is its number of linear extensions.

Lemma 7.4. *Let $\mathbf{n}$ be any content, $E = \{(a, t) : a \in [k], t \in [n_a]\}$, (a, t) standing for the t -th letter a , with the chains $C_a = \{(a, 1), \dots, (a, n_a)\}$, and let $\text{pos}_w(a, t)$ be the position of the t -th letter a of $w \in \mathcal{W}(\mathbf{n})$. Then $w \mapsto \text{pos}_w$ is a bijection onto the bijections $E \rightarrow [| \mathbf{n} |]$ that increase along every chain, and w has the projections $\Lambda = (A^{(1)}, \dots, A^{(k-1)})$ if and only if pos_w satisfies every relation of*

$$R_\Lambda = \{(a, t) \prec (a, t+1)\} \cup \{(i, r) \prec (i+1, s) : r \leq A_s^{(i)}\} \cup \{(i+1, s) \prec (i, r) : r > A_s^{(i)}\},$$

where σ satisfies $x \prec y$ if $\sigma(x) < \sigma(y)$. So the trace class of Λ has e_{R_Λ} words.

Proof. The inverse map puts the letter a at the positions $\sigma(a, 1) < \dots < \sigma(a, n_a)$. As $\text{pos}_w(i, \cdot)$ increases, $A_s^{(i)}(w) = \#\{r : \text{pos}_w(i, r) < \text{pos}_w(i+1, s)\}$ counts an initial segment of $[n_i]$; so $A^{(i)}(w) = A^{(i)}$ if and only if $\text{pos}_w(i, r) < \text{pos}_w(i+1, s) \Leftrightarrow r \leq A_s^{(i)}$ for all r, s . $\square$

Lemma 7.5 ([40, Corollary 4.2, p. 15]). *Let R be an irreflexive relation on a finite set E , and $\mathcal{O}_R = \{\mathbf{u} \in [0, 1]^E : u_x \leq u_y \text{ for every } (x, y) \in R\}$. Then $\text{vol } \mathcal{O}_R = e_R/|E|!$.*

For posets this is Stanley's $e(P) = |P|! \text{vol } \mathcal{O}(P)$; the proof needs only irreflexivity.

Proof. Points with two equal coordinates form a null set. For $\mathbf{u}$ with distinct coordinates let $\sigma_{\mathbf{u}} : E \rightarrow [|E|]$ be its rank map. The $|E|!$ sets $\{\mathbf{u} : \sigma_{\mathbf{u}} = \sigma\}$ are disjoint open simplices, permuted by the permutations of the coordinates, so each has volume $1/|E|!$. For $(x, y) \in R$, $x \neq y$, so $u_x \leq u_y$ if and only if $\sigma_{\mathbf{u}}(x) < \sigma_{\mathbf{u}}(y)$; up to a null set, $\mathcal{O}_R$ is the union of the simplices of the e_R bijections that satisfy R . $\square$

For $v \in [0, 1]$ and $0 \leq p \leq n$ let $h_n(v) = v^{n-1}/(n-1)!$, $H_n(v) = v^n/n!$ and $\vartheta_p(v) = v^p(1-v)^{2n-1-p}/(p!(2n-1-p)!)$; let $\mathcal{K}(v, v') = h_n(1-v')H_n(\min(v, v'))$, with the operator $(\mathcal{K}g)(v) = \int_0^1 \mathcal{K}(v, v')g(v')dv'$, and put $g_2 = 1$, $g_3 = H_n$ and $g_k = \mathcal{K}g_{k-2}$ for $k \geq 4$.

Proposition 7.6. *For all $n \geq 1$, $k \geq 2$ and $A \in \mathcal{A}_1$, $|Z_k(\{A\})| = (nk)! \int_0^1 \vartheta_{A_1}(v) g_k(v) dv$.*

Proof. Let Λ be A with the projections of Z_k on the other edges. As R_Λ is irreflexive, Lemmas 7.4 and 7.5 give $|Z_k(\{A\})| = (nk)! \text{vol } \mathcal{O}_{R_\Lambda}$. On $C_1 \cup C_2$ the relations compare every two elements and are realised by a word, so they amount to $u_{e_1} \leq \dots \leq u_{e_{2n}}$ for a total order $e_1 \prec \dots \prec e_{2n}$, with $(2, 1) = e_{p+1}$, $p = A_1$. Given the chains, a full edge $i \geq 2$ gives only $u_{i,n} \leq u_{i+1,1}$, an empty one only $u_{i+1,n} \leq u_{i,1}$; so $C_1 \cup C_2$ meets the other chains only in $u_{3,n} \leq u_{2,1}$. By Fubini, this volume is the integral of $G_k(u_{2,1})$ over $u_{e_1} \leq \dots \leq u_{e_{2n}}$, where $G_2 = 1$ and, for $k \geq 3$, $G_k(v)$ is the volume of the set $Y_k(v)$ of points of $[0, 1]^{C_3 \cup \dots \cup C_k}$ with the relations among $C_3, \dots, C_k$ and $u_{3,n} \leq v$. Fixing $u_{e_{p+1}} = v$ leaves increasing tuples in $[0, v]$ and $[v, 1]$, of volumes $v^p/p!$ and $(1-v)^{2n-1-p}/(2n-1-p)!$; so the volume is $\int_0^1 \vartheta_p G_k$, and it remains to show $G_k = g_k$. First, $G_3(v) = v^n/n! = H_n(v)$. For $k \geq 4$, the section of $Y_k(v)$ at $u_{4,1} = v'$ is the product of $\{u_{3,1} \leq \dots \leq u_{3,n} \leq \min(v, v')\}$, of volume $H_n(\min(v, v'))$, of $\{v' \leq u_{4,2} \leq \dots \leq u_{4,n} \leq 1\}$, of volume $h_n(1-v')$, and of a copy of $Y_{k-2}(v')$ under $a \mapsto a-2$, which keeps edge parities (a point if $k=4$). So $G_k = \mathcal{K}G_{k-2}$, and $G_k = g_k$ by induction. $\square$

Corollary 7.7. *Let $\vartheta_{\mathcal{S}} = \sum_{A \in \mathcal{S}} \vartheta_{A_1}$. (i) $|Z_k(\mathcal{S})| = (nk)! \int_0^1 \vartheta_{\mathcal{S}} g_k$. (ii) $|Z_k(\{A\})|$ depends only on A_1 . (iii) $|Z_3(\{A\})| = \binom{n+A_1}{n}$; in particular $|Z_3| = \binom{2n}{n}$. (iv) $|Z_2(\{A\})| = 1$, and $r_2(\mathcal{S}) = |\mathcal{S}|$; $|Z_2(\mathcal{S}_u)|$ is the Mann–Whitney frequency of u [18, §4, p. 51].*

Proof. (i) and (ii) follow from Proposition 7.6; $\mathcal{S} = \{(n^n)\}$ gives $|Z_k|$. (iii) $(3n)! \int_0^1 v^{p+n}(1-v)^{2n-1-p} dv / (p!(2n-1-p)!n!) = \binom{p+n}{n}$ by the Beta integral. (iv) $\int_0^1 \vartheta_p = 1/(2n)!$. $\square$

For $k = 3$ and $u \leq n$, (iii) gives $|Z_3(\mathcal{S}_u)|$ as the number of partitions of u if $u < n$ (then $A_1 = 0$ on $\mathcal{S}_u$), and that number plus n if $u = n$; by reversal these are sizes of fibres of (n, n, n) given by [49, Theorem 2, eq. (3.1), p. 5], a trivial overlap.

Lemma 7.8. *For every $k \geq 2$, g_k is a polynomial with rational coefficients of exact degree $n(k-2)$; it is positive on $(0, 1]$, and $g_k(0) = 0$ for $k \geq 3$.*

Proof. Splitting at $v' = v$, $(\mathcal{K}g)(v) = \int_0^v h_n(1-t)H_n(t)g(t)dt + H_n(v) \int_v^1 h_n(1-t)g(t)dt$, a polynomial if g is one. If g has degree d and leading coefficient $\alpha \neq 0$, then, writing $\int_v^1 = \int_0^1 - \int_0^v$, the coefficient of v^{2n+d} in $\mathcal{K}g$ is

$$\frac{\alpha(-1)^{n-1}}{(n-1)!n!} \left(\frac{1}{2n+d} - \frac{1}{n+d} \right) = \frac{\alpha(-1)^n n}{(n-1)!n!(2n+d)(n+d)} \neq 0,$$

and all other terms have lower degree; so $\deg \mathcal{K}g = \deg g + 2n$, which with $\deg g_2 = 0$ and $\deg g_3 = n$ gives $\deg g_k = n(k-2)$. As $\mathcal{K}(v, v') > 0$ for $v \in (0, 1]$ and $v' \in (0, 1)$, $g > 0$ on $(0, 1)$ implies $\mathcal{K}g > 0$ on $(0, 1]$; and $H_n(\min(0, v')) = 0$ gives $(\mathcal{K}g)(0) = 0$. $\square$

The transfer is an instance of the transfer-operator method for volumes of order polytopes, used for alternating permutations [9, eq. (37), Lemma 3], Markovian polytopes [3, §3, Definition 7, Proposition 8, eqs. (18)–(19), arXiv pp. 12–13], glued bipartite posets [6, Theorem 1.1, Example 2.5] and densities of descent words [7, Proposition 2.1, eqs. (2.1)–(2.3), p. 152], cf. [8, §3]; for $n = 1$, $\mathcal{K}(v, v') = \min(v, v')$ composes the recurrences (2.2) and (2.1) of [7] for the uniform density, and the chains add only the weights h_n, H_n .

7.3 The likelihood-ratio order and the parity sandwich

Let $I = (0, 1)$ and let $\mathcal{P}$ be the set of bounded continuous functions $I \rightarrow (0, \infty)$; by Lemma 7.8 it contains every g_k (restricted to I), and it contains H_n .

Definition 7.9. For $f, h \in \mathcal{P}$, $f \leq_{\text{lr}} h$ if h/f is nondecreasing on I , that is, if $f(s)h(t) - f(t)h(s) \geq 0$ for all $s < t$ in I ; the comparison is *strict* if h/f is not constant.

Lemma 7.10. *The relation $\leq_{\text{lr}}$ is reflexive and transitive on $\mathcal{P}$.*

Proof. $h/h = 1$, and a product of positive nondecreasing functions is nondecreasing. $\square$

Lemma 7.11. *For $v_1 < v_2$ and $s < t$ in $[0, 1]$, $\mathcal{K}(v_1, s)\mathcal{K}(v_2, t) \geq \mathcal{K}(v_1, t)\mathcal{K}(v_2, s)$.*

Proof. The factor $h_n(1-s)h_n(1-t)$ is common to both sides, and $H_n \geq 0$ is nondecreasing. If $v_1 \leq s$, both sides have the factor $H_n(v_1)$, and the claim reads $H_n(\min(v_2, t)) \geq H_n(\min(v_2, s))$; if $s < v_1$, both have $H_n(s)$, and it reads $H_n(\min(v_2, t)) \geq H_n(\min(v_1, t))$. Both hold, as $\min(v_2, t)$ is at least $\min(v_2, s)$ and $\min(v_1, t)$. $\square$

Lemma 7.12. *If $f, h \in \mathcal{P}$, then $\mathcal{K}f, \mathcal{K}h \in \mathcal{P}$, and $f \leq_{\text{lr}} h$ implies $\mathcal{K}f \leq_{\text{lr}} \mathcal{K}h$.*

Proof. $\mathcal{K}f$ is bounded and continuous, as $\mathcal{K}$ is continuous on $[0, 1]^2$, and positive on I as in Lemma 7.8. For $v_1 < v_2$ in I let $\mathcal{D}(s, t) = \mathcal{K}(v_1, s)\mathcal{K}(v_2, t) - \mathcal{K}(v_2, s)\mathcal{K}(v_1, t)$, so that $\mathcal{D}(t, s) = -\mathcal{D}(s, t)$. The left side below is $\iint_{I^2} f(s)h(t)\mathcal{D}(s, t) ds dt$; exchanging the variables on $\{s > t\}$, the diagonal being a null set,

$$(\mathcal{K}f)(v_1)(\mathcal{K}h)(v_2) - (\mathcal{K}f)(v_2)(\mathcal{K}h)(v_1) = \iint_{s < t} (f(s)h(t) - f(t)h(s))\mathcal{D}(s, t) ds dt,$$

which is nonnegative by Definition 7.9 and Lemma 7.11. $\square$

Lemma 7.13. *For every $f \in \mathcal{P}$, $1 \leq_{\text{lr}} \mathcal{K}f \leq_{\text{lr}} H_n$.*

Proof. As $\mathcal{K}(v, v')$ is nondecreasing in v , so is $\mathcal{K}f$. On I , $(\mathcal{K}f)(v)/H_n(v)$ is nonincreasing, since so is $\mathcal{K}(v, v')/H_n(v) = \min(1, v'/v)^n h_n(1 - v')$ for every v' . $\square$

Lemma 7.14. *Let $\chi : I \rightarrow (0, \infty)$ be continuous, strictly decreasing, with $\vartheta_n \chi$ integrable, and $\mathcal{Q}(g) = \int_I \vartheta_n \chi g / \int_I \vartheta_n g$. Then $g \leq_{\text{lr}} h$ implies $\mathcal{Q}(g) \geq \mathcal{Q}(h)$, strictly unless h/g is constant.*

Proof. The integrals converge, and $\int_I \vartheta_n g > 0$. Exchanging s and t and averaging,

$$\int_I \vartheta_n \chi g \int_I \vartheta_n h - \int_I \vartheta_n \chi h \int_I \vartheta_n g = \frac{1}{2} \iint_{I^2} \vartheta_n(s)\vartheta_n(t)(\chi(s) - \chi(t))(g(s)h(t) - h(s)g(t)) ds dt,$$

and the two brackets have the same sign, so the difference is nonnegative and $\mathcal{Q}(g) \geq \mathcal{Q}(h)$. If $(h/g)(s_0) < (h/g)(t_0)$ for some $s_0 < t_0$, then by continuity $(h/g)(s) < (h/g)(t)$ on a product $U \times V$ of intervals around s_0 and t_0 with $U < V$, where the integrand is positive, as $g(s)h(t) - h(s)g(t) = g(s)g(t)((h/g)(t) - (h/g)(s))$; so the difference is positive. $\square$

These are standard total-positivity arguments ($\mathcal{K}$ is totally positive of order two), given in full as we need them on the open interval, with strictness; for $n = 1$ the proof of [7, Theorem 3.5, p. 155] integrates a product of two such 2×2 determinants over $[0, 1]^2$, as here.

Proof of Theorem H(ii). By Lemma 7.13, $g_2 \leq_{\text{lr}} g_k \leq_{\text{lr}} g_3$ for $k \geq 4$, as $g_k = \mathcal{K}g_{k-2}$; this holds for $k = 2, 3$ too, $g_3/g_2 = H_n$ being nondecreasing. Applying $\mathcal{K}$ $i - 1$ times (Lemma 7.12), $g_{2i} \leq_{\text{lr}} g_{k+2i-2} \leq_{\text{lr}} g_{2i+1}$ for all $i \geq 1$ and $k \geq 2$, and $k + 2i - 2$ runs over all indices $\geq 2i$. By Corollary 7.7, $r_k(\mathcal{S}) = \mathcal{Q}(g_k)$ with $\chi = \vartheta_{\mathcal{S}}/\vartheta_n$, and by direct division $\vartheta_p(v)/\vartheta_n(v) = c_p((1-v)/v)^{n-p}$ with $c_p = \frac{n!(n-1)!}{p!(2n-1-p)!}$. So $\chi = \sum_{A \in \mathcal{S}} c_{A_1}((1-v)/v)^{n-A_1}$ is a nonempty sum of positive, continuous, strictly decreasing functions on I , as $n - A_1 \geq 1$ for non-full A , and $\vartheta_n \chi = \vartheta_{\mathcal{S}}$ is integrable. Lemma 7.14 gives $r_{2i+1}(\mathcal{S}) \leq r_k(\mathcal{S}) \leq r_{2i}(\mathcal{S})$ for every $k \geq 2i$. If $k \neq 2i$, then g_k and g_{2i} have different degrees (Lemma 7.8), so g_k/g_{2i} is not constant on I and the right inequality is strict; likewise the left one if $k \neq 2i + 1$.

Taking $k = 2i + 2$ and $k = 2i + 3$ gives $r_{2i+2} < r_{2i}$ and $r_{2i+1} < r_{2i+3}$. For $i, j \geq 1$, the first part with the index j and $k = 2i + 1$ if $i \geq j$, or with the index i and $k = 2j$ if $i < j$, gives $r_{2i+1} < r_{2j}$. So both subsequences are monotone and bounded, and converge; $r_{2i+1} < r_{2i+3} \leq r_{\infty}^{\text{odd}}(\mathcal{S})$, $r_{\infty}^{\text{even}}(\mathcal{S}) \leq r_{2j+2} < r_{2j}$, and the limit of $r_{2i+1} < r_{2j}$ is $r_{\infty}^{\text{odd}}(\mathcal{S}) \leq r_{\infty}^{\text{even}}(\mathcal{S})$. $\square$

Corollary 7.15. *Let $k \geq 3$, $X \in \{Z_k, Z'_k\}$, $e \in \{1, k - 1\}$, and let $X[e \leftarrow u]$ be the one-end relaxation in which the value of $\psi_2(X)$ at e is replaced by u . Then $|X[e \leftarrow u]| = |Z_k(\mathcal{S}_{u^*})|$, with $u^* = u$ if X has the value n^2 at e and $u^* = n^2 - u$ if it has the value 0; so $u^* \leq n^2 - 1$.*

Proof. The mirror map turns a relaxation at edge $k - 1$ into one at edge 1 with the same area and the same alternating value there (Lemma 2.3). At edge 1, a relaxation of Z_k with area u is $Z_k(\mathcal{S}_u)$, and reversal, which maps the fibre of $\mathbf{x}$ onto that of $(n^2 - x_i)_i$, turns a relaxation of Z'_k with area u into $Z_k(\mathcal{S}_{n^2-u})$. $\square$

Remark 7.16 (Permutations). For $n = 1$ the only non-full projection is (0) ; $w \in Z_k$ if and only if $u = w^{-1}$ is alternating, $u(1) < u(2) > u(3) < \dots$, and $w \in Z_k(\{(0)\})$ if and only if $u(1) > u(2) > u(3) < u(4) > \dots$, a defect at one end. By [5, Corollary 3.1, p. 428] with $a = 0$, $|Z_k(\{(0)\})| = kE_{k-1} - E_k$ (the sequence [24]), so $r_k = kE_{k-1}/E_k - 1$, which tends to $\pi/2 - 1$ [5, Theorem 3.2, p. 427]. With $E_m = 2(2/\pi)^{m+1}m!\sigma_m$, $\sigma_m = \sum_{i \text{ odd}} ((-1)^{(i-1)/2}i)^{-m-1}$ [5, eq. (2.4), p. 424], $r_k = (\pi/2)\sigma_{k-1}/\sigma_k - 1$. Here $\sigma_m > 1$ decreases over odd m , and $\sigma_m = 1 - \sum_{j \geq 1} ((4j-1)^{-s} - (4j+1)^{-s}) < 1$, $s = m+1$, increases over even m , as $a^{-s} - b^{-s}$ decreases in $s \geq 1$ for $3 \leq a < b$ ($(b/a)^s \geq b/a > \ln b / \ln a$). So r_k increases over odd k below $\pi/2 - 1$ and decreases over even k above it: for $n = 1$, Theorem H(ii) is a direct consequence of [5], with $r_\infty^{\text{odd}} = r_\infty^{\text{even}} = \pi/2 - 1$. This end-defect limit differs from the constant $2/\pi$ of [5, Proposition 4.2, p. 429] for a defect inside the pattern, which obeys a parity sandwich of the same shape [5, Proposition 4.1, p. 428], from inequalities of [8] (see [8, Figure 5, p. 14]); for $n \geq 2$ we located none of this.

7.4 Consequences for the largest classes

Let $r_3(n) = r_3(\mathcal{S}_{n^2-n})$, the ratio of the class of edge-1 deficit n .

Lemma 7.17. $r_3(n) = \sum_{\kappa \vdash n} \binom{2n-\kappa_1}{n} / \binom{2n}{n}$, where κ_1 is the largest part of κ .

Proof. By Corollary 7.7(iii), $r_3(\mathcal{S}) = \sum_{A \in \mathcal{S}} \binom{n+A_1}{n} / \binom{2n}{n}$. For $A \in \mathcal{S}_{n^2-n}$ the numbers $n - A_t$ decrease weakly and sum to n ; without zeros they form a partition κ of n with $\kappa_1 = n - A_1$, and every partition of n arises once, as $A_t = n - \kappa_t$ ($\kappa_t = 0$ beyond its length). $\square$

Proposition 7.18. $r_3(n) > 1$ if and only if $n \geq 4$: $r_3(1) = 1/2$, $r_3(2) = 2/3$, $r_3(3) = 3/4$, $r_3(4) = 71/70$ and $r_3(5) = 41/36$; the $\kappa_1 \leq 2$ terms alone exceed 1 if and only if $n \geq 6$.

Proof. The partitions with $\kappa_1 \leq 2$ are (1^n) and $(2^j, 1^{n-2j})$, $1 \leq j \leq n/2$. As $\binom{2n-1}{n} / \binom{2n}{n} = 1/2$ and $\binom{2n-2}{n} / \binom{2n}{n} = (n-1)/(2(2n-1))$, they give $L(n) = 1/2 + \lfloor n/2 \rfloor (n-1)/(2(2n-1))$; $L(n) > 1$ means $\lfloor n/2 \rfloor (n-1) > 2n-1$, that is, $2h^2 - 5h + 1 > 0$ for $n = 2h$ and $2h^2 - 4h - 1 > 0$ for $n = 2h + 1$; both hold exactly for $h \geq 3$. So $L(n) > 1$ exactly for $n \geq 6$ ($L(4) = 13/14$, $L(5) = 17/18$), and the other terms are positive. For $n \leq 5$ the terms $\binom{2n-\kappa_1}{n}$ (κ in reverse lexicographic order) are 1; 1, 3; 1, 4, 10; 1, 5, 15, 15, 35; 1, 6, 21, 21, 56, 56, 126; divided by $\binom{2n}{n} = 2, 6, 20, 70, 252$, their sums are $1/2, 2/3, 3/4, 71/70$ and $287/252 = 41/36$. $\square$

Corollary 7.19. If $n \geq 4$ and $k \geq 3$, then $|Z_k(\mathcal{S}_{n^2-n})| > |Z_k| = |Z'_k|$; as $Z_k(\mathcal{S}_{n^2-n})$ is one Parikh class, no alternating class of (n^k) is a largest Parikh class.

Proof. By Theorem H(ii) and Proposition 7.18, $r_k(\mathcal{S}_{n^2-n}) \geq r_3(n) > 1$, with no computation. $\square$

Corollary 7.20. For every $k \geq 4$, no alternating class of (3^k) is a largest Parikh class.

Proof. As $r_3(3) = 3/4$, we need exact values. Let $\mathcal{S}_6 = \{(0, 3, 3), (1, 2, 3), (2, 2, 2)\}$ (deficit 3). We use four exact values: $|Z_7| = 1\,942\,824\,800$ and $|Z_7(\mathcal{S}_6)| = 1\,962\,796\,290$, so $r_7(\mathcal{S}_6) > 1$; and, in (3^5) , $|Z_5| = 57\,400$, while the ψ_2 -fibre of $(6, 0, 9, 3)$ has 66\,024 words. They were computed from Proposition 7.6 in exact arithmetic and recomputed by discrete counts sharing no code with it (`checks/check_lr_sandwich.py`, sections [4], [8], [10d]). By Theorem H(ii), $r_k(\mathcal{S}_6) > r_7(\mathcal{S}_6)$ for even $k \geq 4$ (e.g. $|Z_4(\mathcal{S}_6)| = 798 > |Z_4| = 662$) and $r_k(\mathcal{S}_6) \geq r_7(\mathcal{S}_6)$ for odd $k \geq 7$, so $Z_k(\mathcal{S}_6)$ is larger than Z_k and Z'_k . For $k = 5$, as $r_5(\mathcal{S}_6) = 55\,020/57\,400 < 1$, take the two-end fibre of $(6, 0, 9, 3)$: it agrees with $\psi_2(Z_5)$ outside the non-adjacent edges 1 and 4, so it is one Parikh class (Corollary 3.3), larger than Z_5 and Z'_5 . $\square$

Proposition 7.21. *Every one-end relaxation of an alternating class of (3^5) , at either end and with any area at the relaxed edge, has fewer than $|Z_5| = 57\,400$ words.*

Proof. Such a relaxation has $|Z_5(\mathcal{S}_u)|$ words (Corollary 7.15). By Proposition 7.6 and two discrete counts (section [10d] of `checks/check_lr_sandwich.py`) these numbers are 4410, 4410, 8820, 24 780, 24 780, 36 330, 55 020, 50 610, 34 650 ($u = 0, \dots, 8$), all below 57 400. $\square$

Remark 7.22 (verified by computer). The fibre of $(6, 0, 9, 3)$ is the largest of the fibres $(u, 0, 9, u')$ of (3^5) , Z_5 included; it and its mirror $(3, 9, 0, 6)$ are single classes. For $3 \leq k \leq 40$, $r_k(\mathcal{S}_u)$ is largest, among $0 \leq u \leq 8$, exactly at $u = 6$: the best one-end relaxation of (3^k) has deficit 3 (section [10d] of `checks/check_lr_sandwich.py`; $k \leq 20$ by default, $k \leq 40$ with `--full`). This is not proved for all k ; the largest classes of (n^k) , $n \geq 3$, are unknown.

Proposition 7.23 (verified by computer). *The 1680 words of (3^3) form 218 Parikh classes. The largest have $20 = |Z_3| = \binom{6}{3}$ words, and exactly six classes attain it: Z_3 , Z'_3 and the classes $(x, y, z) = (6, 3, 0)$, $(6, 3, 6)$, $(3, 6, 0)$, $(3, 6, 6)$.*

All words were enumerated (sections [8], [10d] of `checks/check_lr_sandwich.py`; [P] of `checks/check_article2_data.py`).

Corollary 7.24. *Let $n = 2$ and $k \geq 3$. Every one-end relaxation of an alternating class of (2^k) , at either end and with any area at the relaxed edge, has at most $\frac{50}{53}|Z_k|$ words, with equality only for $k = 4$ and area 2 at the relaxed edge.*

Proof. By Corollary 7.15, where area 2 at the relaxed edge means $u^* = 2$, it suffices to show $r_k(\mathcal{S}_u) \leq 50/53$ for $0 \leq u \leq 3$, with equality only for $(k, u) = (4, 2)$. The projections of area 0, 1, 2, 3 are $(0, 0)$; $(0, 1)$; $(0, 2)$ and $(1, 1)$; $(1, 2)$. By Corollary 7.7(ii) and the exact values $|Z_4(\{(0, 0)\})| = 15$, $|Z_4(\{(1, 1)\})| = 35$ and $|Z_4| = 53$, computed from Proposition 7.6 and recounted by enumeration (sections [3] and [10e] of `checks/check_lr_sandwich.py`), $|Z_4(\mathcal{S}_u)| = 15, 15, 50, 35$ for $u = 0, 1, 2, 3$. By Theorem H(ii), $r_k(\mathcal{S}_u) < r_4(\mathcal{S}_u)$ for $k = 3$ and for $k \geq 5$. So $r_k(\mathcal{S}_u) \leq r_4(\mathcal{S}_u) \leq 50/53$, with equality only for $(k, u) = (4, 2)$. $\square$

Remark 7.25 (Two-end relaxations; verified by computer). Corollary 7.24 is about one-end relaxations, and this matters at $k = 3$: the central fibre $(2, 2)$ of $(2, 2, 2)$ relaxes both ends of Z_3 and Z'_3 , and its two classes, $(x, y, z) = (2, 2, 0)$ and $(2, 2, 2)$, have $6 = |Z_3|$ words each, a tie. For $4 \leq k \leq 12$, over all pairs of end areas and both alternating classes, the largest two-end relaxation of (2^k) has area 2 at both ends and fewer words than Z_k , for instance $28 < 53$ for $k = 4$ and $680 < 840$ for $k = 5$ (section [10e] of `checks/check_lr_sandwich.py`; $k \leq 9$ by default, $k \leq 12$ with `--full`); for $k \geq 4$ these fibres are single classes (Corollary 3.3).

Proof of Theorem H(iii). Combine Corollaries 7.19, 7.20 and 7.24 ($n \geq 4$, $n = 3$, $n = 2$). $\square$

Figure 8 shows the three regimes, and finitely many exact values bracket the parity limits:

Corollary 7.26. *For every $\mathcal{S}$ and $i, j \geq 1$, both parity limits lie in $(r_{2i+1}(\mathcal{S}), r_{2j}(\mathcal{S}))$. In particular, for $n = 3$ and $\mathcal{S} = \mathcal{S}_6$ they lie in $(r_{15}, r_{14}) \subset (1.026442860, 1.026593038)$ and, using the values up to $k = 30$, in $(r_{29}, r_{30}) \subset (1.0264917187, 1.0264917215)$; for $n = 2$ and $\mathcal{S} = \mathcal{S}_2$ they lie in $(r_{59}, r_{60}) \subset (0.843915414, 0.843915415)$.*

Proof. The first statement is Theorem H(ii). The endpoints are exact rationals, computed from Proposition 7.6 and recounted by inserting the letter chains one at a time; the decimal lower endpoints are truncated and the upper ones rounded up (section [10f] of `checks/check_lr_sandwich.py`, and section [U] of `checks/check_article2_data.py`). $\square$

Question 7.27. *For $n \geq 2$ and every $\mathcal{S}$ as in Theorem H(ii), is $r_\infty^{\text{odd}}(\mathcal{S}) = r_\infty^{\text{even}}(\mathcal{S})$?*

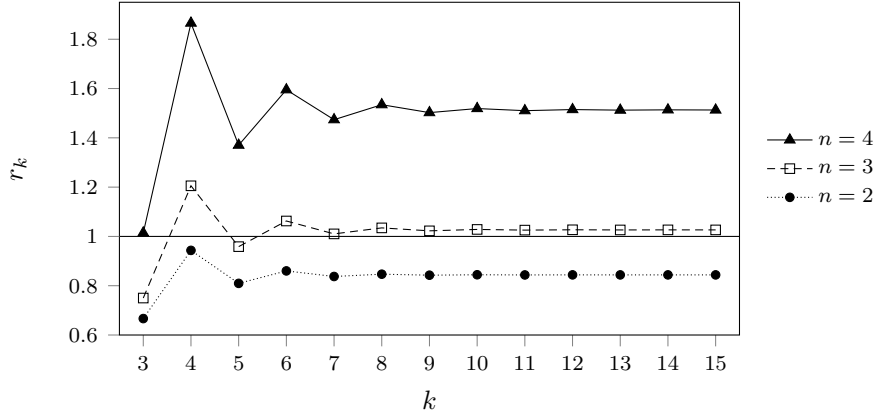


Figure 8: The one-end ratios $r_k = |Z_k(\mathcal{S})|/|Z_k|$ of the uniform contents (n^k) , where $\mathcal{S} = \mathcal{S}_{n^2-n}$ is the set of edge-1 projections of area $n^2 - n$ (edge-1 deficit n), for $n = 2, 3, 4$ and $3 \leq k \leq 15$; the solid line is $r_k = 1$. The odd-indexed values increase, the even-indexed values decrease, and each odd-indexed value lies below each even-indexed one (Theorem H). For $n = 2$, $r_k \leq r_4 = 50/53$ for every $k \geq 3$; for $n = 3$, $r_3, r_5 < 1 < r_k$ for $k = 4$ and $6 \leq k \leq 15$; for $n = 4$, $r_k > 1$ from $r_3 = 71/70$ on. At $k = 2$ the ratios are 2, 3 and 5.

For $n = 1$ the answer is yes, with the end-defect limit $\pi/2 - 1$ (Remark 7.16). For $n \geq 2$ we conjecture no direction, although the brackets close fast, and know no closed form; a spectral argument for the compact operator $\mathcal{K}$ is a natural route, not attempted here (Section 9).

7.5 Two occurrences of each letter

Proposition 7.28 (verified by computer). (i) In (2^3) the largest Parikh classes are Z_3 , Z'_3 , $\Theta_\emptyset$ and $\Theta_{\{1\}}$, with 6 words each. (ii) In (2^4) the largest classes are $\Theta_{\{1\}}$ and $\Theta_{\{2\}}$, exchanged by the mirror map, with 56 words each, while $|Z_4| = 53$. (iii) In (2^5) the largest class is $\Theta_{\{2\}}$, with 844 words, while $|Z_5| = 840$; the next class of the central fibre has 698 words. (iv) For $6 \leq k \leq 12$, Z_k and Z'_k are the only largest Parikh classes of (2^k) , and the second largest are exactly the four one-end relaxations with area 2 at the relaxed edge.

Sections [3], [6] of `checks/check_r7_switch.py` give (i)–(iii) by enumerating all words, as does section [P] of `checks/check_article2_data.py`. For (iv), its section [C1] computes every ψ_2 -fibre and splits into classes each fibre with an entry 2 and at least $|Z_k(\mathcal{S}_2)|$ words, the others being single trace classes (Lemma 2.6); $k \leq 9$ by default, $k \leq 12$ with `--full`; its section [PM], an independent computation of all classes, confirms $k \leq 9$ with `--full`.

Conjecture 7.29. For $k \geq 6$, Z_k and Z'_k are the only largest Parikh classes of (2^k) , and the second largest are exactly the four one-end relaxations with area 2 at the relaxed edge.

By Proposition 7.28 this holds for $6 \leq k \leq 12$, and by Corollary 7.24 the conjectured second largest classes lose to Z_k for every $k \geq 3$, with the ratio $r_k(\mathcal{S}_2) \leq 50/53$, whose parity limits lie in $(0.843915414, 0.843915415)$; a bound for the other classes is missing. For permutations the alternating maximum and its uniqueness follow from a dominance order on descent lists, refined by the last letter [8, §3, Definition 3.4, Corollary 3.8, Theorem 3.10, pp. 8–9]; an analogue for (2^k) is left to a later article.

Conjecture 7.30. Let $k \geq 4$ and $n_i \in \{1, 2\}$ for every i . Then $Z(\mathbf{n})$ and $Z'(\mathbf{n})$ are the only largest Parikh classes of $\mathbf{n}$ if and only if $\mathbf{n}$ does not begin with $(2, 2, 1)$, does not end with $(1, 2, 2)$, and is neither (2^4) nor (2^5) .

Table 4: The checks, the statements they recompute, and wall-clock times in seconds of logged runs: default, a passing run of the whole suite on 21 October 2026 (279.7s in all) and, for `check_article2.py`, one of 22 October; `--full`, runs of 20 to 23 October, in five and three parts for `check_article2_data.py` and `check_article2.py`.

Check	Statements	Default	--full
<code>check_rigidity.py</code>	Lemma 3.2, Theorem A, Corollaries 3.4, 3.6 and 3.7	28.1	260.9
<code>check_r2_extrema.py</code>	Theorem B; Corollaries 4.2 to 4.4; Example 4.5	13.0	170.7
<code>check_top_gap.py</code>	Theorem C and Proposition 4.6, hence the converse in Theorem A(i); the top of a fibre in Section 4	11.0	137.5
<code>check_antidiagonal.py</code>	Theorem D; Remark 5.4; Proposition 5.5	5.7	63.3
<code>check_2b2.py</code>	Theorem E, its lemmas and the enumeration for $b \leq 19$ in its proof; Proposition 5.7 and Corollary 5.8; Table 3	13.2	331.0
<code>check_r7_golden.py</code>	Theorem F and its lemmas; the certificate of Appendix A	17.4	–
<code>check_h1_centre.py</code>	Theorem G and the lemmas of its proof; Corollary 6.11	12.4	221.2
<code>check_r7_switch.py</code>	Lemma 3.1 and Corollary 3.3; Theorem H(i); Propositions 7.3 and 7.28; Conjecture 7.29 for $k \leq 7$, Conjecture 7.30 for $k \leq 5$	21.0	–
<code>check_lr_sandwich.py</code>	Theorem H(ii)–(iii), its lemmas and the exact values of its proof; Remarks 7.16 and 7.25	11.6	146.3
<code>check_article2_data.py</code>	a second recomputation of the quoted data and figures; with <code>--full</code> , Conjecture 7.29 for $k \leq 12$, Conjecture 7.30 for $k \leq 11$ and Question 5.6 for $a + b + c \leq 45$	20.5	1431.6
<code>check_article2.py</code>	every number, formula and range printed, from its registry; the figure data	24.9	1570.4
<i>Further recomputations, written during the exploration</i>			
<code>check_r5_opening.py</code>	the case (1^k) ; the largest trace classes of small contents	18.8	–
<code>check_r7_ternary.py</code>	Proposition 5.5; Question 5.6 for $a + b + c \leq 30$	18.1	–
<code>check_r7_mechanism.py</code>	Theorem D(ii); Proposition 5.5	2.5	–
<code>check_r7_general.py</code>	data on (2^k) and (3^k) for small k (Section 7)	10.2	–
<code>check_ternary_defect.py</code>	Propositions 4.10 and 4.13	0.2	–

The ‘only if’ direction holds, by Proposition 7.3 and, for (2^4) and (2^5) , Proposition 7.28; the ‘if’ direction holds without two adjacent 2s (Remark 7.1) and contains Conjecture 7.29. Section [C4] of `checks/check_article2_data.py` verified it for all $\{1, 2\}$ -contents with $4 \leq k \leq 11$, one of each pair $\mathbf{n}, \mathbf{n}^{\text{rev}}$ ($k \leq 8$ by default, $k \leq 11$ with `--full`; section [PM] recomputes $k \leq 8$ from all classes), splitting every fibre with at least $|Z(\mathbf{n})|$ words that is not one trace class. Of the 2130 contents, exactly 491 have a class with more words than $Z(\mathbf{n})$, namely the excepted ones; in the others the alternating classes are the only largest ones, with no tie.

8 Verification

Every computational statement of this article (the numbers in the text, tables and figures, the ranges of the statements verified by computer and the finite computations used in proofs) is recomputed by a deterministic Python 3.11 check using only the standard library (Table 4); the checks are distributed with the article, in the verification package on its web page and in the repository <https://github.com/cyamba/yamba-ai-math/tree/main/research/parikh>. Counts and the finite computations in proofs are exact, in integer or rational arithmetic, and where practical a quantity is computed by two independent methods. For a proved statement a check is a test on finite ranges; for a statement verified by computer, and for a finite computation used in a proof, it is the certificate. Each check runs on its own, with `--full` for the larger ranges quoted here, and exits with a non-zero status on any failure. The times in Table 4 are process data from the logs of named runs on a shared machine, not recomputed, and so indicative only.

Proofs and computation. All proofs are by hand except three, which contain an exact finite computation, repeated by the named check: the cases $b \leq 19$ of Theorem E, with Corollary 5.8 (sections [H], [J] of `checks/check_2b2.py`); Lemma 6.9, hence Theorem F, by the certificate of Appendix A (section [6] of `checks/check_r7_golden.py`); and Theorem H(iii) for $n = 2, 3$, with Proposition 7.21 and Corollary 7.26 (sections [3], [4], [8], [9], [10d]–[10f] of `checks/check_lr_sandwich.py`). The ‘only if’ direction of Conjecture 7.30 also uses Proposition 7.28. Statements resting on computation alone are marked as verified by computer and established on the stated ranges only; Conjecture 7.29 for $10 \leq k \leq 12$ and Conjecture 7.30 for $9 \leq k \leq 11$ rest on one method, and for $k \leq 9$ and $k \leq 8$ respectively on two.

9 Open problems

Three letters. For every b the words $2^b u$, with u of content (a, c) over $\{1, 3\}$, form the fibre $(0, bc)$, a single Parikh class of $\tau = \binom{a+c}{a}$ words (cf. [34, pp. 110–111], where $b = 0$ and $a = c$); as a word of a trace class is determined by its subword on $\{1, 3\}$ (Lemma 2.4), no trace class is larger and $d_{\max}(a, b, c) \geq \tau$; Theorems F and G raise the following question.

Question 9.1. *Let $b \geq 3$.*

- (i) *For which (a, c) is no Parikh class of (a, b, c) larger than the largest trace class, that is, $d_{\max}(a, b, c) = \binom{a+c}{a}$?*
- (ii) *For which (a, c) is the centre $(ab/2, bc/2)$ the unique largest ψ_2 -fibre of (a, b, c) ?*

For $b = 1$ every (a, c) has the property in (i), as the content is isolated and its classes are trace classes (Theorem A(iii), Lemma 2.6), and for $b = 2$, $a \leq c$, the answer is $c(c-1) \leq a(a+c-1)$ (Theorem F). For $b \geq 3$ the answer to (i) contains $(1, 1)$, as $(1, b, 1)$ is isolated and its largest fibres have $\tau = 2$ words (Theorem D, Remark 5.4), but not $(2, 2)$, as $d_{\max}(2, b, 2) > 6 = \tau$ (Proposition 5.7, Theorem E); and while some largest class of $(a, 2, c)$ lies on the boundary of the box (Theorem F), none of $(2, b, 2)$ does for $b \geq 20$ (Theorem E(ii)). In (ii), reversal maps the fibre of (x, y) onto that of $(ab-x, bc-y)$ (Lemma 2.3), so a unique largest fibre is the centre, with ab and bc even. For $b = 2$ the answer is every $(a, c) \neq (1, 1)$ (Theorem G), and for every b it contains $(2, 2)$ exactly if b is even (Theorem D(iii)); for other contents with b even the centre can lose (Proposition 5.5), and Question 5.6 asks where the largest fibres lie and when the centre is the unique one; for odd $b \geq 3$ we have no conjecture. Beyond the contacts listed in Table 2 we do not take up the further directions of [49, §7].

Contents with multiplicities 1 and 2. Conjecture 7.29 is verified for $6 \leq k \leq 12$ (Proposition 7.28); one-end relaxations of the alternating classes lose for every $k \geq 3$ (Corollary 7.24), and two-end relaxations tie with them for $k = 3$ and are verified to lose for $4 \leq k \leq 12$ (Remark 7.25). By Iriarte’s theorem (Remark 7.1), any class other than Z_k and Z'_k with at least $|Z_k|$ words merges several trace classes, so it lies in a fibre with some $x_i = 2$, as the areas 0, 1, 3, 4 of an edge determine its projection (Lemma 2.6); the central fibre alone splits into F_k classes (Theorem H(i)). A bound for such merged classes is what is missing. One route is an analogue of the dominance argument of [8, §3] for permutations: compare the cumulative counts of the position of the last letter in each class of (2^m) with those in an alternating class, and carry these comparisons from m to $m+1$ across extreme edges, where classes factorise (Lemma 3.1); as they include the sizes, the induction cannot start below $m = 6$, by the cases $k = 4, 5$ of Proposition 7.28, and only a strict form gives uniqueness. Conjecture 7.30, which contains Conjecture 7.29, is open only in its ‘if’ direction and only for contents with two adjacent letters that occur twice (Remark 7.1); it is verified for $4 \leq k \leq 11$.

End defects. With $\mathcal{S}_u$ as in Section 7, every one-end relaxation of an alternating class of (n^k) has the size of a $Z_k(\mathcal{S}_u)$, $0 \leq u < n^2$ (Corollary 7.15), and $r_k(\mathcal{S}_u)$ has parity limits $r_\infty^{\text{odd}}(\mathcal{S}_u) \leq r_\infty^{\text{even}}(\mathcal{S}_u)$ (Theorem H(ii)). If both ends could be relaxed independently, the best gain for k of one parity would be the square of the largest such limit, or 1 if none exceeds 1.

Question 9.2. *Let $n \geq 2$. As $k \rightarrow \infty$ through the odd integers, does $d_{\max}(n^k)/|Z_k|$ tend to $\max(1, \max_u r_\infty^{\text{odd}}(\mathcal{S}_u))^2$, and through the even integers to $\max(1, \max_u r_\infty^{\text{even}}(\mathcal{S}_u))^2$?*

For $n = 2$ these limits are at most $50/53$ (Corollary 7.24), so a positive answer would give $d_{\max}(2^k) \sim |Z_k|$, an asymptotic form of Conjecture 7.29. For $n \geq 3$ some limit exceeds 1: for $n \geq 4$ as $r_3(\mathcal{S}_{n^2-n}) > 1$ (Proposition 7.18) and the odd ratios increase, and for $n = 3$ by Corollary 7.26. For permutations both limits equal $\pi/2 - 1$ (Remark 7.16), and Ehrenborg proved product rules of this kind: permutations of $[k]$ whose descent set is alternating except for fixed patterns at the two ends number asymptotically E_k times the product of one constant for each end [5, Theorems 5.2 and 5.3, p. 432], and he conjectures a general one [5, Conjecture 7.1, p. 435]; the constant $2/\pi$ of [5, Proposition 4.2, p. 429] is the limit for an interior defect. A spectral analysis of the operator $\mathcal{K}$ of Section 7 is a natural approach to Question 9.2 and to Question 7.27.

A programme. Rigidity is a condition on pairs of consecutive edges (Theorem A(ii)), so rigid vectors are counted by walks, with rational generating functions for periodic multiplicities (Theorem A(iv)); is there one for all contents at once? For $r \geq 4$, when do the diagonals $3, \dots, r$ determine the Parikh matrix at a given content, and is the answer again local? Globally, “no matter how many diagonals we fix, the Parikh matrix is still not necessarily determined” [31, p. 228, and Theorem 4]. Likewise, Section 7 counts the alternating classes of (n^k) and their one-end relaxations by iterating one integral operator with a totally positive kernel. Does this extend to every periodic sequence of multiplicities, with kernels depending on the multiplicities of consecutive letters, and do the ratios of end relaxations then obey sandwiches of the same kind? Are the alternating classes of a periodic sequence the largest for all large k exactly when every one-end limit is below 1? For uniform contents this holds for $n = 1$, classically, and for $n \geq 3$ by Theorem H; for $n = 2$ it would follow from Conjecture 7.29.

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A Proof of the Cauchy-surface lemma

We prove Lemma 6.9 in the notation of Section 6, writing K for $K_{a,c}$.

Reduction. By (6.2) we may assume that μ is the smallest of μ , $a - \mu$, ν , $c - \nu$:

$$\mu \leq a - \mu, \quad \mu \leq \nu, \quad \mu \leq c - \nu. \quad (\text{A.1})$$

Three cases remain. (I) μ is a half-integer: if ν is one too, then $K = 0$ by Lemma 6.2(i); otherwise $\nu \geq \mu + \frac{1}{2}$ is an integer, and the complement turns $K_{a,c}(\mu, \nu)$ into $K_{c,a}(\nu, \mu)$, with the integer column ν and the half-integer height $\mu < \nu$ (Lemma A.3). (II) $\mu = 1$ (Lemma A.4). (III) $\mu \geq 2$ is an integer, and $a - \mu \geq \mu \geq 2$ (Lemma A.10).

The column basis. Fix an integer μ with $1 \leq \mu \leq a - 1$, and put $\alpha = a - \mu$. Let $\ell_j = \binom{\alpha+c-j}{\alpha}$ for $0 \leq j \leq c$, the number of paths from (μ, j) to (a, c) , and $\ell_j = 0$ for $j > c$. Then $\ell_j \geq \ell_{j+1}$ and $\nabla^2 \ell_j = \ell_j - 2\ell_{j+1} + \ell_{j+2}$ is $\binom{\alpha-2+c-j}{\alpha-2}$ for $j \leq c-2$, $\alpha-1$ for $j = c-1$, 1 for $j = c$ and 0 for $j > c$; so $\nabla^2 \ell_j \geq 0$, and $\nabla^2 \ell_j > 0$ for $j \leq c$ if $\alpha \geq 2$. Let $u(n) = \binom{\mu+n}{\mu}$ and $U(n) = \binom{\mu+n+1}{\mu+1}$ for $n \geq 0$, and 0 for $n < 0$; both are nondecreasing and convex on $\mathbb{Z}$, and $\sum_{k \leq n} u(k) = U(n)$. For a sequence (e_j) let $E_1(J) = \sum_{j \leq J} e_j$ and $E_2(J) = \sum_{J' \leq J} E_1(J')$.

Lemma A.1. *Let (μ, ν) be an interior centre with $\mu \in \mathbb{Z}$, and $0 \leq m \leq 2c$. Then*

$$P_0(m) = \sum_{j \geq 0} p_j^{(m)} \ell_j, \quad p_j^{(m)} = \sum_{i=\lceil m/2 \rceil}^{\min(m,j)} \binom{\mu-1+j-i}{\mu-1}, \quad (\text{A.2})$$

$$K^\vee(\mu, \nu) = \sum_{i=\lceil \nu \rceil}^{2\nu} u(2\nu - i) \ell_i, \quad (\text{A.3})$$

$$K^h(\mu, \nu) \leq \binom{\mu+\nu}{\mu-1} (\ell_\nu - \ell_{\nu+1}) \quad \text{if } \nu \in \mathbb{Z}. \quad (\text{A.4})$$

Proof. A path from $(0, i)$ to (a, c) first reaches the column μ at a height $j \geq i$, coming from $(\mu - 1, j)$; summing $\binom{\mu-1+j-i}{\mu-1} \ell_j$ over j and over $\lceil m/2 \rceil \leq i \leq \min(m, c)$ gives (A.2) by Proposition 6.3. In $K^\vee$, $u(2\nu - i)$ paths reach the lower point $(\mu, 2\nu - i)$ and ℓ_i continue from the upper point (μ, i) . In K^h , the term $j = \mu + s$, $1 \leq s \leq \mu$, is $\binom{\mu-s+\nu}{\nu}$ times the number of paths from $(\mu + s, \nu)$, at most the number $\ell_\nu - \ell_{\nu+1}$ of paths from $(\mu + 1, \nu)$ (those from (μ, ν) whose first step is east); and $\sum_{s=1}^{\mu} \binom{\mu-s+\nu}{\nu} = \binom{\mu+\nu}{\mu-1}$. $\square$

Let $K^+(\mu, \nu)$ be the right side of (A.3), plus that of (A.4) if $\nu \in \mathbb{Z}$; so $K \leq K^+$, and the coefficients of $P_0(m)$ and K^+ in the basis (ℓ_j) depend on μ, ν, m only.

Lemma A.2. $\sum_j e_j \ell_j = \sum_J (\ell_J - \ell_{J+1}) E_1(J) = \sum_J \nabla^2 \ell_J E_2(J)$ for every sequence (e_j) . Hence $\sum_j e_j \ell_j \geq 0$ if $E_1 \geq 0$ or $E_2 \geq 0$; for the coefficients of $P_0(m) - K^+$ this gives $P_0(m) \geq K^+ \geq K$ for all contents with $a > \mu$ and $c > \nu$ at once.

Proof. Two summations by parts ($\ell_J = 0$ for $J > c$); the weights are nonnegative. $\square$

Lemma A.3. *Let $\mu \geq 1$ be an integer and $\nu = l + \frac{1}{2}$ with $0 \leq l \leq \mu - 1$. Then $K(\mu, \nu) < \tau$ for all contents with $a \geq \mu + 1$ and $c \geq l + 1$.*

Proof. Here $K = K^\vee$. The coefficients $\binom{\mu-1+j}{\mu-1}$ of $\tau = P_0(0)$ have the prefix sums $u(J)$; those of K , $\sum_{i=l+1}^J u(2l+1-i)$, vanish for $J \leq l$ and never exceed $U(l) = \binom{\mu+l+1}{\mu+1} < \binom{\mu+l+1}{\mu} = u(l+1) \leq u(J)$ for $J \geq l+1$, the quotient being $(\mu+1)/(l+1) > 1$. So $E_1 \geq 0$ and $E_1(c) > 0$, with weight $\ell_c - \ell_{c+1} = 1$, and Lemma A.2 gives $\tau - K \geq E_1(c) > 0$. $\square$

Lemma A.4. *Let $a \geq 2$ and $1 \leq \nu < c$, $\nu \in \frac{1}{2}\mathbb{Z}$. (i) $K(1, 1) = 3 \binom{a+c-2}{a-1} \leq \tau$, with equality if and only if $a = c = 2$. (ii) $K(1, 2) \leq P_0(2)$, with equality if and only if $(a, c) = (2, 3)$. (iii) $K(1, \nu) < P_0(2 \lfloor \nu/2 \rfloor)$ for integers $\nu \geq 3$. (iv) $K(1, \nu) < P_0(2 \lceil \nu/2 \rceil)$ for $\nu = l + \frac{1}{2}$, $l \geq 1$.*

Proof. Here $\mu = 1$, and by (A.2) $p_j^{(2h)} = \min(j - h, h) + 1$ for $j \geq h$ and 0 for $j < h$. (i) By Lemma 6.2(iii), $K(1, 1) = 2 \binom{a+c-2}{a-1} + \binom{a+c-3}{a-1} + \binom{a+c-3}{a-2} = 3 \binom{a+c-2}{a-1}$, and $\tau / \binom{a+c-2}{a-1} = (a+c)(a+c-1)/(ac) \geq 3$ if and only if $a^2 - ac + c^2 \geq a+c$, true as $a^2 - ac + c^2 \geq ac \geq a+c$, with equality if and only if $a = c = 2$. For integers $\nu \geq 2$ let $h = \lfloor \nu/2 \rfloor$. Then $K^h(1, \nu) = \binom{a+c-\nu-2}{a-2} = \ell_\nu - \ell_{\nu+1}$, so $K(1, \nu) = \sum_{t=0}^{\min(\nu, c-\nu)} (\nu-t+1) \ell_{\nu+t} + \ell_\nu - \ell_{\nu+1}$, with the coefficients $\nu+2$ at ν , $\nu-1$ at $\nu+1$, and $2\nu-j+1$ at $\nu+2 \leq j \leq \nu + \min(\nu, c-\nu)$; let e_j be $p_j^{(2h)}$ minus the coefficient of ℓ_j in $K(1, \nu)$. (ii) For $\nu = 2$, $h = 1$: $e_1 = 1$, $e_2 = -2$, $e_3 = 1$, $e_4 = 1$ (if $c \geq 4$), $e_j = 2$ for $5 \leq j \leq c$, and $e_j = 0$ otherwise. So $E_1 = 0, 1, -1, 0$ and $E_2 = 0, 1, 0, 0$ for $J = 0, \dots, 3$, both are at least 1 for $4 \leq J \leq c$, and Lemma A.2 gives $P_0(2) \geq K(1, 2)$. If

$a \geq 3$, $\nabla^2 \ell_1 > 0$ meets $E_2(1) = 1$ and the inequality is strict; if $a = 2$, $\ell_j = c - j + 1$ is linear and $P_0(2) - K(1, 2) = E_2(c)$, which is 0 for $c = 3$ and positive for $c \geq 4$.

(iii) Let $\nu \geq 3$. For $J < \nu$ only $p^{(2h)}$ contributes: $E_1(J) = 0$ for $J < h$ and $E_1(J) \geq 1$ for $h \leq J < \nu$. For $J = \nu + S$, $0 \leq S \leq \min(\nu, c - \nu)$, the prefix sums of $p^{(2h)}$ are $\sum_{t=0}^{J-h} (\min(t, h) + 1)$ and those of $K(1, \nu)$ are $\nu + 2$ at $S = 0$ and $(S + 1)(\nu + 1) - S(S + 1)/2$ for $S \geq 1$. For $\nu = 2h$ this gives $E_1(\nu) = (h + 1)(h - 2)/2$ and $E_1(\nu + S) = (S - h)(S - h + 1)/2$ ($S \geq 1$); for $\nu = 2h + 1$, $E_1(\nu) = (h + 2)(h - 1)/2$ and $E_1(\nu + S) = (S - h)(S - h - 1)/2$; all are nonnegative. For larger J (when $c > 2\nu$) the prefix sums of $K(1, \nu)$ are constant, so $E_1(J) \geq E_1(2\nu) \geq 0$. By Lemma A.2, $P_0(2h) - K(1, \nu) \geq E_1(h)(\ell_h - \ell_{h+1}) = \binom{a+c-h-2}{a-2} > 0$.

(iv) Let $h = \lceil l/2 \rceil$, so $2h \leq l + 1 \leq c$; $K(1, \nu) = \sum_{s=0}^{\min(l, c-l-1)} (l - s + 1)\ell_{l+1+s}$. For $J \leq l$, $E_1(J) \geq 0$. For $J = l + 1 + S$, $0 \leq S \leq \min(l, c - l - 1)$, $2E_1(J)$ is $S^2 - (2h - 3)S + h^2 - h + 2$ if $l = 2h - 1$ and $S^2 - (2h - 1)S + h^2 + h + 2$ if $l = 2h$, with negative discriminants $1 - 8h$ and $-8h - 7$; so $E_1(J) > 0$, and $E_1(J) \geq E_1(2l + 1) > 0$ for larger J . With the weight $\ell_{l+1} - \ell_{l+2} = \binom{a+c-l-3}{a-2} > 0$ at $J = l + 1$, Lemma A.2 gives $P_0(2h) > K(1, \nu)$. $\square$

Columns $\mu \geq 2$. For integers $\mu \geq 1$, $\nu \geq \mu$ and $h \geq 0$ consider the conditions

$$u(\nu - h) + u(\nu - h - 1) \leq u(\nu + 1), \quad (\text{C-ii})$$

$$\sum_{k=\nu-2h}^{\nu-h} U(k) \geq \binom{\mu+\nu+1}{\mu}, \quad (\text{C-a})$$

$$\sum_{k=\nu-h-1}^{\nu-1} U(k) \geq \sum_{k=\nu-h}^{\nu-1} (U(\nu) - U(k)) + \binom{\mu+\nu}{\mu-1}, \quad (\text{C-b})$$

$$\sum_{k=\nu-h-1}^{\nu-1} U(k) \geq \sum_{k=\nu-h}^{\nu-1} (U(\nu) - U(k)), \quad (\text{C-b'})$$

so that (C-b') is (C-b) without its last term, and let $h_{\min}(\mu, \nu)$ be the least $h \geq 0$ with (C-ii); it is at most ν , as at $h = \nu$ the left side of (C-ii) is 1.

Lemma A.5. *Let $\mu \geq 1$, $\nu \geq \mu$ and $0 \leq h \leq \nu$ be integers, $a \geq \mu + 1$ and $c \geq \nu + 1$. (i) If (C-ii), (C-a) and (C-b) hold at h , then $K(\mu, \nu) \leq P_0(2h)$. (ii) If (C-ii) and (C-b') hold at h , then $K(\mu, \nu + \frac{1}{2}) \leq P_0(2h + 1)$. Both are strict if $a \geq \mu + 2$ and $h \leq \nu - 1$.*

Proof. (i) Let e_j be the coefficients of $P_0(2h) - K^+(\mu, \nu)$. For $J \leq \nu - 1$ only $P_0(2h)$ contributes, and by the hockey stick $E_1(J) = \sum_{i=h}^{2h} u(J - i) \geq 0$. For $J \geq \nu - 1$, (A.3) contributes the prefix sums $U(\nu) - U(2\nu - J - 1)$ and (A.4) the value $\binom{\mu+\nu}{\mu-1}$ at $J = \nu$ only; so $E_1(J) = E^\circ(J) - \binom{\mu+\nu}{\mu-1}[J = \nu]$, where $E^\circ(J) = \sum_{i=h}^{2h} u(J - i) - U(\nu) + U(2\nu - J - 1)$ is convex on $[\nu - 1, \infty)$, with $E^\circ(\nu - 1) \geq 0$. The window identity $\sum_{i=h}^{2h} u(\nu + h - i) = U(\nu) - U(\nu - h - 1)$ gives $E^\circ(\nu + h) = 0$, and $E^\circ(\nu + h + 1) = u(\nu + 1) - u(\nu - h) - u(\nu - h - 1) \geq 0$ by (C-ii). So $E^\circ \geq 0$ on $[\nu + h, \infty)$, and $\{E^\circ < 0\}$ is an interval ending at $\nu + h - 1$, possibly empty. Hence $E_2 \geq 0$ on $[0, \nu - 1]$, and as the increments $E_2(J) - E_2(J - 1) = E^\circ(J)$ for $J \geq \nu + 1$ are nonnegative, then negative on that interval, then nonnegative, the minimum of E_2 on $[\nu, \infty)$ is $\min(E_2(\nu), E_2(\nu + h))$. Summing, $E_2(\nu) = \sum_{k=\nu-2h}^{\nu-h} U(k) - \binom{\mu+\nu+1}{\mu}$ and $E_2(\nu + h) = \sum_{k=\nu-h}^{\nu} U(k) - (h + 1)U(\nu) + \sum_{k=\nu-h-1}^{\nu-1} U(k) - \binom{\mu+\nu}{\mu-1}$, the differences of the two sides of (C-a) and of (C-b). So $E_2 \geq 0$, and Lemma A.2 gives $P_0(2h) \geq K^+ \geq K$. (ii) The same argument with $l = \nu$: (A.3) has the coefficients $u(2l + 1 - i)$ for $l + 1 \leq i \leq 2l + 1$, with prefix sums $U(l) - U(2l - J)$ for $J \geq l$, $P_0(2h + 1)$ has the prefix sums $\sum_{i=h+1}^{2h+1} u(J - i)$, and there is no horizontal term. So $E^\circ = E_1$ is convex on $[l, \infty)$, $E^\circ(l + h + 1) = 0$, $E^\circ(l + h + 2) = u(l + 1) - u(l - h) - u(l - h - 1) \geq 0$, and the minimum of E_2 on $[l, \infty)$ is $\min(E_2(l), E_2(l + h + 1))$, where $E_2(l) = \sum_{i=h+1}^{2h+1} U(l - i) \geq 0$ and $E_2(l + h + 1)$ is the difference of the two sides of (C-b'). Strictness: if $a \geq \mu + 2$, then $\alpha \geq 2$ and $\nabla^2 \ell_J > 0$ for $J \leq c$; in (i), $J = h \leq \nu - 1$ lies below the support of K^+ and $E_2(h) = U(0) = 1$; in (ii), $J = h + 1 \leq l$ lies below that of (A.3) and $E_2(h + 1) = U(0) = 1$. $\square$

Lemma A.6. *If $\nu \geq \mu$, then (C-a), (C-b) and the weaker (C-b') hold at $h = 0$.*

Proof. At $h = 0$, (C-b) reads $U(\nu - 1) \geq \binom{\mu+\nu}{\mu-1}$, as does (C-a), $U(\nu) \geq \binom{\mu+\nu+1}{\mu}$, less $u(\nu)$ on both sides; and $U(\nu - 1)/\binom{\mu+\nu}{\mu-1} = \binom{\mu+\nu}{\mu+1}/\binom{\mu+\nu}{\mu-1} = \nu(\nu + 1)/(\mu(\mu + 1)) \geq 1$. $\square$

At $h = 0$, (C-ii) reads $u(\nu - 1) \leq u(\nu + 1) - u(\nu) = \binom{\mu+\nu}{\mu-1}$, that is, $\mu(\mu + \nu) \geq \nu(\nu + 1)$. So $h_{\min}(\mu, \mu) = 0$, and $h_{\min}(\mu, \nu) \geq 1$ implies $\nu(\nu + 1) > \mu(\mu + \nu)$.

Lemma A.7. *If $\mu \geq 3$ and $h_{\min}(\mu, \nu) = 1$, then (C-a) and (C-b) hold at $h = 1$.*

Proof. As $U(\nu) - U(\nu - 1) + \binom{\mu+\nu}{\mu-1} = \binom{\mu+\nu+1}{\mu}$, both conditions read $U(\nu - 2) + U(\nu - 1) \geq \binom{\mu+\nu+1}{\mu}$, that is, $(\nu + 1)\nu(\mu + 2\nu - 1) \geq (\mu + 1)(\mu + \nu + 1)(\mu + \nu)$. As $\nu(\nu + 1) > \mu(\mu + \nu)$, it suffices that $\mu(\mu + 2\nu - 1) \geq (\mu + 1)(\mu + \nu + 1)$, that is, $(\mu - 1)(\nu - 3) \geq 4$; and $\nu(\nu + 1) > \mu(\mu + \nu) \geq 3(3 + \nu)$ gives $\nu \geq 5$, so $(\mu - 1)(\nu - 3) \geq 2 \cdot 2$. $\square$

Lemma A.8. *Let $h = h_{\min}(\mu, \nu) \geq 1$. Then*

$$u(\nu - h + 1) + u(\nu - h) > u(\nu + 1), \quad (\text{A.5})$$

and: (i) $h < (1 - 2^{-1/\mu})(\nu + \mu + 1) \leq (\ln 2)(\nu + \mu + 1)/\mu$; (ii) if $h \geq 2$, then $\nu + 1 > 3\mu$; (iii) if $\mu \geq 3$, or $\mu = 2$ and $\nu \geq 6$, then $h \leq \nu - \mu - 1$ and (C-a) holds.

Proof. (A.5) is the failure of (C-ii) at $h - 1$. (i) By (A.5), $2u(\nu - h + 1) > u(\nu + 1)$, while $u(\nu + 1)/u(\nu + 1 - h) = \prod_{i=1}^h (\nu + 1 + i)/(\nu + 1 - h + i) \geq (1 - h/(\nu + \mu + 1))^{-\mu}$; so $(1 - h/(\nu + \mu + 1))^\mu > \frac{1}{2}$, and $1 - 2^{-1/\mu} \leq (\ln 2)/\mu$. (ii) If $h \geq 2$, then $u(\nu - 1) + u(\nu - 2) > u(\nu + 1)$; with $q = (\nu + 1)/(\mu + \nu + 1)$, $u(\nu - 1)/u(\nu + 1) \leq q^2$ and $u(\nu - 2)/u(\nu + 1) \leq q^3$, so $q^2 + q^3 > 1 > (3/4)^2 + (3/4)^3$, whence $q > 3/4$, that is, $\nu + 1 > 3\mu$. (iii) For $\mu \geq 3$ we have $\nu \geq \mu + 1$, and (C-ii) holds at $h' = \nu - \mu - 1$, as $u(\mu + 1) + u(\mu) \leq u(\mu + 2) \leq u(\nu + 1)$ by $u(\mu + 2) - u(\mu + 1) = \binom{2\mu+1}{\mu-1} \geq \binom{2\mu}{\mu} = u(\mu)$, which holds if and only if $\mu^2 - 2\mu - 2 \geq 0$; so $h \leq \nu - \mu - 1$. For $\mu = 2$ and $\nu \geq 6$, (i) gives $h < (1 - 1/\sqrt{2})(\nu + 3) \leq \nu - 3$. For (C-a), let $n_0 = \nu - h - 1 \geq \mu$. By (A.5), $\binom{\mu+\nu+1}{\mu} = u(\nu + 1) < u(\nu - h + 1) + u(\nu - h)$, and two terms of the left side of (C-a) give $U(\nu - h) + U(n_0) = u(\nu - h) + 2U(n_0)$; so (C-a) follows from $2U(n_0) \geq u(n_0 + 2)$, that is, from $2(n_0 + 1)(n_0 + 2) \geq (\mu + 1)(\mu + n_0 + 2)$, which holds at $n_0 = \mu$ and with an increasing difference for $n_0 \geq \mu$. $\square$

Lemma A.9. *If $\mu \geq 2$, $\nu \geq \mu$ and $h = h_{\min}(\mu, \nu) \geq 2$, then (C-b) holds.*

Proof. A sufficient condition. As $U(\nu) - U(k) = \sum_{j=k+1}^{\nu} u(j)$, (C-b) says $(h + 1)U(\nu) \geq \sum_{r=0}^h (2r + 1)u(\nu - h + r) + \binom{\mu+\nu}{\mu-1}$. Subtract $(h + 1)\sum_{r=0}^h u(\nu - h + r) = (h + 1)(U(\nu) - U(\nu - h - 1))$ and pair the indices s and $h - s$ in $\sum_{s=0}^h (h - 2s)u(\nu - s)$:

$$(\text{C-b}) \iff (h + 1)U(\nu - h - 1) \geq \sum_{0 \leq s < h/2} (h - 2s)[u(\nu - s) - u(\nu - h + s)] + \binom{\mu+\nu}{\mu-1}.$$

Let $u_* = u(\nu - h)$ and $g_j = u(\nu - h + j)/u_*$, an increasing convex sequence, and divide by u_* : $U(\nu - h - 1)/u_* = (\nu - h)/(\mu + 1)$; $g_{h-s} - g_s \leq (h - 2s)(g_h - g_{h-1}) = (h - 2s)g_h\mu/(\mu + \nu)$; $\binom{\mu+\nu}{\mu-1}/u_* = g_h\mu/(\nu + 1)$; $\sum_{0 \leq s < h/2} (h - 2s)^2 = h(h + 1)(h + 2)/6$; and (A.5) says $g_{h+1} < 1 + g_1$, so $g_h < 2 + \mu/(\nu - h + 1)$. Hence (C-b) follows from

$$\frac{\nu - h}{\mu + 1} \geq \left(2 + \frac{\mu}{\nu - h + 1}\right)\mu \left[\frac{h(h + 2)}{6(\mu + \nu)} + \frac{1}{(h + 1)(\nu + 1)}\right]. \quad (\text{A.6})$$

The asymptotic range. Replace $1/((h + 1)(\nu + 1))$ by $1/(3(\nu + 1))$. The left side then decreases and the right side increases in h , so it suffices to take the real number $\eta = 0.6932(1 + t)$ for h , where $t = (\nu + 1)/\mu$, since $h < (\ln 2)(1 + t) < \eta$ by Lemma A.8(i). With $\nu = \mu t - 1$, the left side

is $t - (t+1+\eta)/(\mu+1)$ and the right side is $(2+1/(t-\eta/\mu))[\eta(\eta+2)/(6(1+t-1/\mu))+1/(3t)]$. For $\mu \geq \mu_0$ and $t \geq t_1$ we have $t - (t+1+\eta)/(\mu+1) \geq \lambda_2 t - \lambda_1$, $t - \eta/\mu \geq \vartheta_1$, $(1+t)/(1+t-1/\mu) \leq \kappa_1$ and $1/(3t) \leq 1/(3t_1)$, with $\lambda_1 = 1.6932/(\mu_0 + 1) = 1 - \lambda_2$, $\vartheta_1 = t_1 - 0.6932(1 + t_1)/\mu_0$ and $\kappa_1 = (1 + t_1)/(1 + t_1 - 1/\mu_0)$. If $\vartheta_1 > 0$, the right side is at most $m_2 t + m_1$, with $m_2 = (2+1/\vartheta_1)\kappa_1(0.6932)^2/6$ and $m_1 = (2+1/\vartheta_1)(\kappa_1 \cdot 0.6932 \cdot 2.6932/6 + 1/(3t_1))$, and (A.6) holds for all $t \geq t_1$ once $\lambda_2 > m_2$ and $(\lambda_2 - m_2)t_1 \geq m_1 + \lambda_1$. Exact rational arithmetic confirms all this for $(\mu_0, t_1) = (5, 3)$, $(4, 4)$, $(3, 5)$, $(2, 8)$ (item [C3] below). As $h \geq 2$ forces $t > 3$ (Lemma A.8(ii)), (C-b) holds for all $\mu \geq 5$, for $\mu = 4$ with $\nu \geq 15$, for $\mu = 3$ with $\nu \geq 14$, and for $\mu = 2$ with $\nu \geq 15$. *The finite range.* For $\mu \in \{2, 3, 4\}$ and $\mu \leq \nu \leq 14$, (C-b) at $h_{\min}$ is checked by exact integer arithmetic (item [C1] below). $\square$

Lemma A.10. *Let $\mu \geq 2$ be an integer, $\nu \geq \mu$, and consider contents with $a \geq \mu+1$ and $c > \nu$. (i) If $\nu \in \mathbb{Z}$ and $(\mu, \nu) \neq (2, 3)$, then $K(\mu, \nu) \leq P_0(2h_{\min}(\mu, \nu))$. (ii) $K(2, 3) \leq P_0(0) = \tau$. (iii) If $\nu = l + \frac{1}{2}$, then $K(\mu, \nu) \leq P_0(2h_{\min}(\mu, l) + 1)$. All three are strict when $a \geq \mu + 2$.*

Proof. (i) By Lemma A.5(i) it suffices that (C-a) and (C-b) hold at $h = h_{\min}(\mu, \nu)$. If $h = 0$, this is Lemma A.6; if $h = 1$ and $\mu \geq 3$, Lemma A.7. For $\mu = 2$, $h = 1$ occurs only for $3 \leq \nu \leq 6$, as $u(\nu - 1) + u(\nu - 2) = \nu^2 \leq u(\nu + 1) = (\nu + 2)(\nu + 3)/2$ if and only if $\nu \leq 6$, and $\nu = 4, 5, 6$ are in [C1]. If $h \geq 2$, (C-a) holds by Lemma A.8(iii) (for $\mu = 2$, $h \geq 2$ forces $\nu \geq 7$) and (C-b) by Lemma A.9. (ii) At $(2, 3)$, $h_{\min} = 1$, and (C-a) and (C-b) fail: both read $14 \geq 15$. With $m = 0$, the coefficients of $P_0(0) - K^+(2, 3)$ have $E_1(J) = -5, -1, 2, 8$ for $J = 3, 4, 5, 6$, $E_2(J) = 10, 5, 4, 6$ for $J = 2, \dots, 5$ and $E_2 \geq 0$ ([C2]), so Lemma A.2 concludes. (iii) By Lemma A.5(ii) it suffices that (C-b') holds at $(\mu, l, h_{\min}(\mu, l))$, which (C-b) at (μ, l) implies, by (i), except at $(\mu, l) = (2, 3)$, where it reads $14 \geq 10$ ([C1]). Strictness: Lemma A.5 needs $h \leq \nu - 1$ in (i) and $h_{\min}(\mu, l) \leq l - 1$ in (iii), which is clear for $h = 0$ and for $\mu = 2$, $h = 1$, and follows from Lemma A.8(iii) otherwise; in (ii), $E_2(0) = 1$ meets $\nabla^2 \ell_0 > 0$. $\square$

Proof of Lemma 6.9. Take a representative with (A.1). In case (I), $K = 0$, or Lemma A.3, applied to the content $(c, 2, a)$, gives $K < \tau$. In case (II), $a \geq 2$, and Lemma A.4 gives $K \leq \tau$ or $K \leq P_0(m)$, with equality only for $(a, c) = (2, 2)$, $(\mu, \nu) = (1, 1)$, where $K = \tau$ is the largest boundary fibre (Proposition 6.6 and Corollary 6.5(ii)), and for $(a, c) = (2, 3)$, $(\mu, \nu) = (1, 2)$, where $K = P_0(2) = f(1) < \tau$ by Corollary 6.5(ii). In case (III), $a \geq \mu + 2$ and $c \geq \nu + \mu$, and Lemma A.10 gives $K < P_0(m)$. Each bound is a boundary fibre of $(a, 2, c)$ or $(c, 2, a)$, which the symmetries carry to one of the original content. $\square$

The finite certificate

The proof leaves three finite families of exact inequalities to a computer, the items [C1]–[C3]; their quantities are integers or rational numbers, and Tables 5 to 7 give their values (those of [C3] rounded), so they can also be checked by hand.

[C1] Columns $\mu \in \{2, 3, 4\}$, heights $\nu \leq 14$. For $\mu \in \{2, 3, 4\}$ and every integer ν with $\mu \leq \nu \leq 14$, (C-a) and (C-b) hold at $h = h_{\min}(\mu, \nu)$ unless $(\mu, \nu) = (2, 3)$, and (C-b') holds at h ; these are 71 cases (Table 5). The proof uses only (C-b) at the pairs with $h \geq 2$ outside the asymptotic range of Lemma A.9, which is for $\mu = 4$ with $\nu \geq 15$, for $\mu = 3$ with $\nu \geq 14$ and for $\mu = 2$ with $\nu \geq 15$; (C-a) and (C-b) for $\mu = 2$ and $h = 1$, that is, for $\nu \in \{4, 5, 6\}$; and (C-b') at $(2, 3)$, where it reads $14 \geq 10$. The rest is a consistency check.

[C2] The centre $(2, 3)$. For $(\mu, \nu) = (2, 3)$ and $m = 0$ the coefficients of $P_0(0)$ are $\binom{1+j}{1} = j + 1$, and those of $K^+(2, 3)$ are 15, 1, 3, 1 at $j = 3, 4, 5, 6$ and 0 elsewhere: the vertical coefficients $u(6 - i) = 10, 6, 3, 1$ at $i = 3, \dots, 6$, and the bound for the horizontal part,

Table 5: Item [C1]: $h = h_{\min}(\mu, \nu)$ and the slacks (left side minus right side) of (C-a), (C-b) and (C-b') at h , the last for the centre $(\mu, \nu + \frac{1}{2})$.

ν	h	$\mu = 2$			h	$\mu = 3$			h	$\mu = 4$		
		(C-a)	(C-b)	(C-b')		(C-a)	(C-b)	(C-b')		(C-a)	(C-b)	(C-b')
2	0	0	0	4								
3	1	-1	-1	4	0	0	0	15				
4	1	9	9	15	0	14	14	35	0	0	0	56
5	1	27	27	34	1	21	21	49	0	42	42	126
6	1	55	55	63	1	76	76	112	1	48	48	168
7	2	66	66	75	1	171	171	216	1	219	219	384
8	2	120	124	134	1	320	320	375	1	539	539	759
9	2	194	203	214	1	539	539	605	1	1078	1078	1364
10	3	217	224	236	2	671	670	748	1	1924	1924	2288
11	3	334	357	370	2	1085	1106	1197	1	3185	3185	3640
12	3	484	526	540	2	1651	1702	1807	1	4991	4991	5551
13	4	525	560	575	2	2401	2491	2611	2	6313	6299	6979
14	4	739	814	830	3	2760	2815	2951	2	9683	9808	10624

$\binom{5}{1} = 5$ at $j = 3$ and -5 at $j = 4$. For $J \geq 7$, $e_J = J + 1 > 0$; so, with Table 6, $E_1 > 0$ and E_2 increases from $J = 5$ on, and $E_2 \geq 0$ everywhere, with $E_2(0) = 1$.

[C3] The closing inequalities. For each $(\mu_0, t_1) \in \{(5, 3), (4, 4), (3, 5), (2, 8)\}$, with 0.6932 read as 6932/10000, the rationals λ_1 , λ_2 , ϑ_1 , κ_1 , m_1 , m_2 of Lemma A.9 satisfy $\vartheta_1 > 0$, $\lambda_2 > m_2$ and $(\lambda_2 - m_2)t_1 \geq m_1 + \lambda_1$ (Table 7). The only real constant enters through $\ln 2 < 0.6932$, which follows from $\ln 2 = \sum_{k \geq 1} 1/(k2^k)$ and an exact tail bound.

Table 6: Item [C2]: the coefficients e_J of $P_0(0) - K^+(2, 3)$ in the basis (ℓ_j) of the column $\mu = 2$, and their prefix sums.

J	0	1	2	3	4	5	6	7
e_J	1	2	3	-11	4	3	6	8
$E_1(J)$	1	3	6	-5	-1	2	8	16
$E_2(J)$	1	4	10	5	4	6	14	30

Table 7: Item [C3]: the slope and the value at $t = t_1$ of the linear bound of Lemma A.9, rounded from their exact rational values.

(μ_0, t_1)	$\lambda_2 - m_2$	$(\lambda_2 - m_2)t_1 - m_1 - \lambda_1$
(5, 3)	0.5147	0.2053
(4, 4)	0.4659	0.5719
(3, 5)	0.3836	0.5930
(2, 8)	0.2486	0.6063

The program. The certificate is the script `notes/r7-scripts/proof-golden/lemmaB_certificate.py`, with `a2c_core.py` from the same directory, among the supplementary files of this article (Section 8; Python 3.11, standard library only; nonzero exit status on any failure). Their SHA-256 digests, as used here, are

```
lemmaB_certificate.py 7374f04790bcd3cc74865279eca217cb3500b8ea831d6bbce0d73df925132a34
a2c_core.py           2f7b4603ee3874770a8f8c2284c3e3f4651ff877477e2be6fe3ded846463e5b9
```

The items [C1]–[C3] take less than a millisecond. The checks [C4]–[C6] are not used by the proof: [C4] tests the criterion at $h_{\min}$ on a large range of (μ, ν) , $E_2 \geq 0$ directly on part of it, and the bound of Lemma A.8(i) in floating point, the script's only floating-point step besides printing the decimals of Table 7; [C5] compares $K_{a,c}(\mu, \nu)$, not strictly, with the boundary fibre named in Lemmas A.4 and A.10, for all integer columns μ , heights $\nu \geq \mu$ and contents with $a + c$ up to the third argument; [C6] checks the algebra of the proofs of Lemmas A.5 and A.9. With the arguments 60 1000 70 the script took 16 s (Python 3.11.9, 22 October 2026). Its name, numbering and symbols follow the project notes: `g`, `Gam`, `b_j`,

K_{tilde} , D , D_2 , μ_{lb} , δ_{a1} , κ , A_1 , A_2 , L_1 , L_2 are u , U , ℓ_j , K^+ , E_1 , E_2 , μ_0 , ϑ_1 , κ_1 , m_1 , m_2 , λ_1 , λ_2 , (ii), (a), (b), (b') are (C-ii)–(C-b'), and possibly half-integer coordinates are passed doubled. Section [6] of `checks/check_r7_golden.py`, run by `checks/run_all.py` (Table 4), re-implements [C1]–[C3] independently, with the decimals of Table 7 and the bound $\ln 2 < 0.6932$.

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